

# Synthesis of a subclass of reciprocal $n$ -ports

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## 1 Abstract

Reciprocity is a fundamental concept in electrical network theory. Most real-life devices are inherently reciprocal, therefore understanding and describing the behavior of reciprocal  $n$ -ports has been the topic of many studies and papers. Antireciprocal  $n$ -ports, though less well known, are also an important and closely related class of  $n$ -ports.

In this study, after describing these two concepts in detail, we discuss  $n$ -ports that are both reciprocal and anti-reciprocal at the same time. We present a simple synthesis of all of the  $n$ -ports that have these properties using a minimum number of ideal transformers.

Finally, we present an interconnection of two ideal transformers into a new 2-port that leads to an interesting singularity. We show how the choice of the turning ratios of these ideal transformers can lead to very different results.

## 2 Matrix description of a multiport

In electrical network theory devices are often modelled as  $n$ -ports. An  $n$ -port – or *multiport* if the number  $n$  has no significance – is an abstract network element with  $n$  pairs of terminals (see Figure 1) and  $k$  linearly independent equations describing the voltages and currents of these ports. These equations can be written in the form  $\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{i} = \mathbf{0}$  with  $\mathbf{u}$  and  $\mathbf{i}$  being vectors of height  $n$ , representing the voltages and currents of each port, respectively, and  $\mathbf{A}$  and  $\mathbf{B}$  being  $k \times n$  real matrices. It is important to note, however, that while these matrices uniquely determine the  $n$ -port by describing the relationships between  $\mathbf{u}$  and  $\mathbf{i}$ , the converse is not true, one  $n$ -port has several different matrix descriptions. More precisely, two networks are *equivalent* if they are described by matrices  $(\mathbf{A}_1|\mathbf{B}_1)$  and  $(\mathbf{A}_2|\mathbf{B}_2)$  respectively, and there exist matrices  $\mathbf{S}$  and  $\mathbf{T}$  such that

$$(\mathbf{A}_1|\mathbf{B}_1) = \mathbf{T}(\mathbf{A}_2|\mathbf{B}_2)$$

$$(\mathbf{A}_2|\mathbf{B}_2) = \mathbf{S}(\mathbf{A}_1|\mathbf{B}_1).$$

The rank of an  $n$ -port can be defined as follows:  $r = r(\mathbf{A}|\mathbf{B})$ . This matrix is often denoted by  $\mathbf{M}$ . If the equation  $n = r$  holds true we call the  $n$ -port *ordinary*. In this work the multiports in discussion are considered ordinary unless specified otherwise.

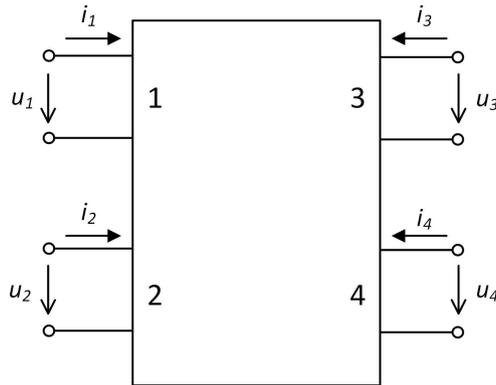


Figure 1: A 4-port

### 3 Various representations of multiports

We saw in Section 2 that multiports are often described by the equation  $\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{i} = \mathbf{0}$ . However, in some special cases, other descriptions can be used to make certain properties of the multiport more apparent. For this, one may consider a slightly different approach to multiport networks. Sometimes it can be beneficial to think of a network as a single multiport and its embedding.

An embedding of a network is *legal* if the rank of the resulting system of equations equals  $2n$ . A legal embedding is *admissible* if it leads to a uniquely solvable network. [1]

If an  $n$ -port has a legal embedding using current sources only, it means that the equations of this  $n$ -port can be written in the form  $\mathbf{u} = \mathbf{R}\mathbf{i}$ . This is called the *resistance*

*description*. Similarly, if there is a legal embedding using only voltage sources then the *conductance description* can be defined as  $\mathbf{i} = \mathbf{G}\mathbf{u}$ . As a generalization of these, if there is an admissible embedding using voltage and current sources only, the  $n$ -port has a *hybrid description*:

$$\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{i}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{D} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{i}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

## 4 Reciprocal and antireciprocal multiports

An ordinary multiport given by the description  $\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{i} = \mathbf{0}$  is *reciprocal*, if  $\mathbf{u}_1^T \mathbf{i}_2 = \mathbf{u}_2^T \mathbf{i}_1$  for any pairs of vectors  $(\mathbf{u}_1, \mathbf{i}_1)$  and  $(\mathbf{u}_2, \mathbf{i}_2)$  satisfying  $\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{i} = \mathbf{0}$ . For various generalizations of this concept in case of non-ordinary multiports, see [2].

Actually, several slightly different definitions of reciprocity exist in the literature of electrical network theory. For a justification why the one used here is the most general and for critical comparison of other appearing definitions, see [3].

A reciprocal multiport always has a hybrid description, and its hybrid matrix looks like this:

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{C}^T & \mathbf{G} \end{bmatrix}$$

where  $\mathbf{R}$  and  $\mathbf{G}$  are symmetrical and the parameters within them have dimensions of resistance and conductance, respectively, while the parameters in the matrix  $\mathbf{C}$  are without dimension [4]. The set of reciprocal multiports is closed with respect to interconnection [5].

Similarly to reciprocity, the term antireciprocity can be defined. An ordinary

multiport given by the description  $\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{i} = \mathbf{0}$  is *antireciprocal*, if  $\mathbf{u}_1^T \mathbf{i}_2 + \mathbf{u}_2^T \mathbf{i}_1 = 0$  for any pairs of vectors  $(\mathbf{u}_1, \mathbf{i}_1)$  and  $(\mathbf{u}_2, \mathbf{i}_2)$  satisfying  $\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{i} = \mathbf{0}$ .

## 5 The inverse and dual of multiports

The term duality often arises in the discussions on electrical networks, and in the past there were multiple interpretations on what the dual of a network is. What most electrical engineers call the dual of a network very different from the dual in the mathematical sense.

The network described by  $(\mathbf{A}_1|\mathbf{B}_1)$  is the *inverse* of the network described by  $(\mathbf{A}_2|\mathbf{B}_2)$  if  $(\mathbf{A}_1|\mathbf{B}_1) \cong (\mathbf{B}_2|\mathbf{A}_2)$  (here the  $\cong$  sign means the two networks are equivalent as described in Section 2). This essentially means that to obtain the inverse of a network one would have to change the roles of the voltages and currents in its representation. Electrical engineers often refer to this new network as the dual of the original, however, from a mathematical point of view it makes more sense to define the dual of a network as follows.

The network described by  $(\mathbf{A}_1|\mathbf{B}_1)$  is the *dual* of the network described by  $(\mathbf{A}_2|\mathbf{B}_2)$  if  $\mathbf{u}_1^T \mathbf{u}_2 + \mathbf{i}_1^T \mathbf{i}_2 = 0$  for any pairs of  $(\mathbf{u}_1, \mathbf{i}_1)$  and  $(\mathbf{u}_2, \mathbf{i}_2)$  satisfying  $(\mathbf{A}_1|\mathbf{B}_1) = 0$  and  $(\mathbf{A}_2|\mathbf{B}_2) = 0$  respectively.

Also, the network described by  $(\mathbf{A}_1|\mathbf{B}_1)$  is the *negative* of the network described by  $(\mathbf{A}_2|\mathbf{B}_2)$  if  $(\mathbf{A}_1|\mathbf{B}_1) \cong (\mathbf{A}_2|-\mathbf{B}_2)$ .

Reciprocity and antireciprocity are closely related to the above terms. The dual of an n-port  $N$  is the negative of the inverse of  $N$  if and only if  $N$  is reciprocal, and the dual equals the inverse if and only if the n-port is antireciprocal. This theorem and the above difference between the dual and the inverse of an n-port was discovered in [6] and it gives us a very important connection between concepts that might be unrelated at first glance.

## 6 Describing interconnections with matroids

Multiports can be interconnected along a graph, to form another multiport. An example of this is presented on Figure 2.

Matroids are a very convenient way to describe this interconnection of multiports, since both the describing matrices of individual multiports, and the graph of their interconnection define a matroid. Consider some multiports interconnected along the graph  $G$ , to form a multiport  $M$ . Let  $\mathcal{G}$  be the direct sum of the cycle matroid on the set of edges corresponding to currents of  $G$ , and the cocycle matroid on the set of edges corresponding to voltages of  $G$ . Let  $\mathcal{A}$  be the direct sum of the matroids of the interconnected multiports. Now if we contract the edges corresponding to the voltages and currents of the original multiports – thereby eliminating the internal variables – in  $\mathcal{G} \vee \mathcal{A}$ , we obtain the matroid  $\mathcal{M}$  of  $M$  if the weaker genericity assumption [7] holds, which means that among the nonzero entries of the matrices of the multiports, the only possible algebraic relations are the ones reflected by the structure of the matroid modelling the multiport.

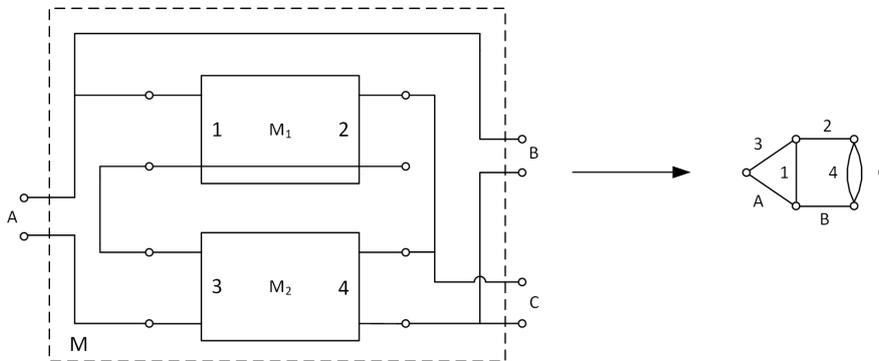


Figure 2: Two 2-ports interconnected to form a 3-port, and the graph of the interconnection

If we consider an embedding of this multiport, and define the matroid  $\mathcal{A}'$  as the direct sum of  $\mathcal{A}$  and all the matroids of the 1-ports of the embedding, then we can obtain a necessary (also sufficient, if the genericity assumption holds) condition on the unique

solvability of this network:  $\mathcal{G} \vee \mathcal{A}'$  has to be the free matroid. Notice that this is only a reformulation of the fact that we need  $2n$  linearly independent equations to solve a system of linear equations with  $2n$  unknowns. However, this reformulation is really useful, since now we can use the matroid partition algorithm [8] to determine – in polynomial time – if a particular embedding is admissible.

Also if the resulting multiport is ordinary, i.e.  $r(\mathcal{M}) = n$ , then it has a hybrid description if, and only if  $\mathcal{G} \vee \mathcal{A} \vee \mathcal{B}_n$  is the free matroid, where  $\mathcal{B}_n = (E, F)$  is a matroid on the set of the currents and voltages of the resulting multiport, where a set  $X \subseteq E$  is independent if, and only if  $|X \cap \{u_k, i_k\}| \leq 1$  for every  $k$  [9]. This is apparent from the fact that the system is uniquely solvable if and only if  $\mathcal{G} \vee \mathcal{A}'$  is the free matroid, as it just means that when we obtained  $\mathcal{A}'$  from  $\mathcal{A}$ , we only added the matroids of current and voltage sources.

## 7 n-ports that are reciprocal and antireciprocal at the same time

In the previous section we saw that reciprocal and antireciprocal networks are two very important and closely related classes of n-ports. Therefore it might be interesting to consider networks that are reciprocal and antireciprocal at the same time. From the definition of reciprocity and antireciprocity it is apparent that since these n-ports have to satisfy the conditions presented in Section 4,  $\mathbf{u}_1^T \mathbf{i}_2$  and  $\mathbf{u}_2^T \mathbf{i}_1$  must be zero for any pairs of vectors  $(\mathbf{u}_1, \mathbf{i}_1)$  and  $(\mathbf{u}_2, \mathbf{i}_2)$  satisfying the equations describing the n-port. This means that the voltages of the ports must be independent from the currents of the ports, and vice versa. Thus, the hybrid matrix of such an n-ports looks like this:

$$\mathbf{H} = \begin{bmatrix} \mathbf{0} & \mathbf{C} \\ -\mathbf{C}^T & \mathbf{0} \end{bmatrix}$$

Notice, that the voltages and the currents are independent from each other in all of these networks. In other words their matroid can be obtained with the formula  $\mathcal{M} = \mathcal{M}_u \oplus \mathcal{M}_i$ , where  $\mathcal{M}_u$  is the matroid describing the relation between the voltages of the network and  $\mathcal{M}_i$  is the matroid describing the relations between the currents. Therefore when analyzing such a network the matroids of its voltages and currents can be studied independently from each other.

The only 2-port that is reciprocal and antireciprocal at the same time is the ideal transformer. As shown in [3] all antireciprocal networks can be realized with transformers and gyrators. However, the ones that are reciprocal as well can be realized in a more simple way using ideal transformers only. If  $\mathbf{C}$  is a  $p \times q$  matrix, then this synthesis can be made with the same amount of transformers as the number of nonzero entries in  $\mathbf{C}$  which is at most  $p \times q$ . If for example the voltage of port  $m$  depends on the voltage of other ports, we just need to connect a transformer with an appropriate ratio to every one of these ports and connect their other sides in series to port  $m$ . An example of this is shown on Figure 3. On the left side are the ports that have current sources attached to them in an admissible embedding, while the ports on the top have voltage sources attached. In total there are  $p \times q$  transformers and the  $i$ th transformer in the  $j$ th column of the figure corresponds to  $c_{ij}$  of  $\mathbf{C}$ . This construction is minimal since one transformer can introduce only one independent variable.

Since the classes of both reciprocal and antireciprocal n-ports are closed with respect to interconnection, interconnecting n-ports that have both of these properties would also lead to a reciprocal and antireciprocal multiport in the general case [6]. However, it might still be interesting to observe what happens, if two transformers are interconnected along the interconnection graph  $K_4$  to form a new 2-port.

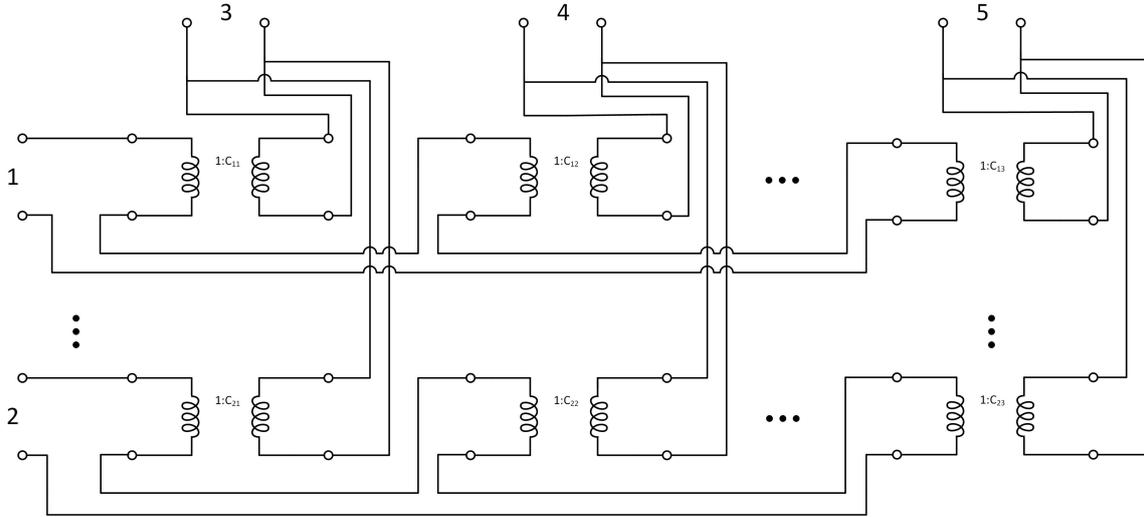


Figure 3: Synthesis construction

Ports are connected in series if their current is the same and the sum of their voltages equals the voltage across their interconnection. They are connected in parallel if their voltage is the same, and the current through their interconnection is equal to the sum of the currents of the ports. As the synthesis uses only series and parallel connections the interconnection graph  $K_4$  is worth investigating since it is neither parallel nor serial - in fact it is the smallest of such connection graphs.

In the general case this interconnection leads to an ordinary ideal transformer. If we consider the ratios of the two transformers interconnected as variables  $k$  and  $j$  and solve the system of equations we obtain the following formula for this new transformers ratio:  $\frac{k+j}{kj+1}$ . We can plot this surface as shown on Figure 4. We can observe from the plot or the formula that aside from the general case when the 2-port is an ideal transformer if  $k+j=0$  or  $kj+1$  the ratio becomes zero and infinity, respectively. This means that in these cases the 2-port behaves like a short circuit open circuit pair.

A more interesting phenomenon arises if the above stated equations both hold true at the same time. There are two possible cases:

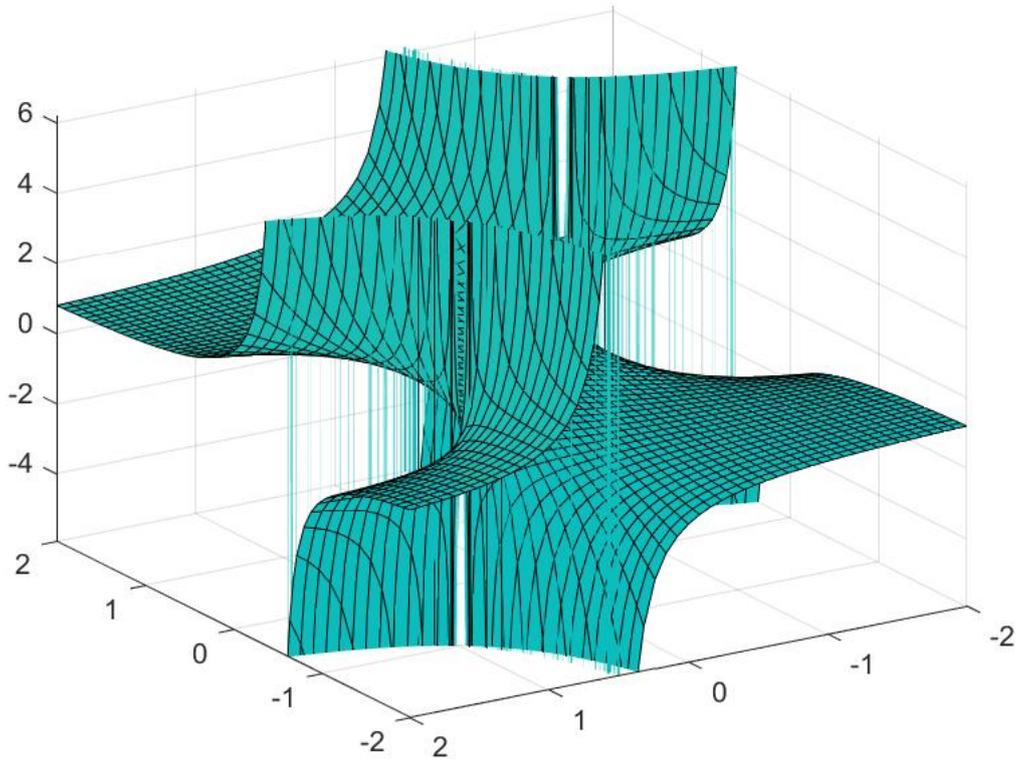


Figure 4: The ratio of the resulting transformer

1.  $k = 1$  and  $j = -1$
2.  $k = -1$  and  $j = 1$

The 2-port is described by its twelve parameters, six voltages and six currents. Since the relations between the voltages and the currents are symmetrical (i.e.  $\mathcal{M}_u$  and  $\mathcal{M}_i$  are the same) and the graph  $K_4$  is the dual of itself therefore its cycle and cocycle matroid is the same, we only have to consider one of them to fully understand the behavior of the resulting 2-port. In general, the equations describing the voltages can be written in the

following form:

$$\begin{bmatrix} 0 & 0 & -k & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -j & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \mathbf{0}$$

The reduced row echelon form of the above matrix in the first case is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

This means, that the inner structure of this network determines that the voltages of both ports are zero. Since, as mentioned above, the relations between the voltages and currents are symmetrical the currents of these ports must be zero too. Therefore in this first case the 2-port becomes a pair of nullators.

In the second case, following the same steps, we obtain the reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that in this case the rank of these equations is four and  $u_1$  and  $u_2$  are independent from each other. Therefore these two quantities are dependent only on the embedding of this 2-port. As the same holds for the currents as well in this case the network becomes a pair of norators.

Notice, that the rank of the resulting 2-port is two in the general case, four in the first case, while in the second case it is zero. This might be surprising considering the fact that the rank of the system of describing equations is ten in the general and also in the first case and eight in the second. So despite the fact that the difference in the ranks of the describing equations between the cases is two, this still leads to a difference of four between the ranks of the resulting 2-ports.

To illustrate why this happens it might be worth considering this 2-port together with its admissible embedding. In the first case since the describing equations uniquely determine the voltages and currents of both ports, the only admissible embedding is a pair of norators. This results in a network which is only terminally-solvable since the port voltages and currents are determined but the internal four voltages and currents are described by three-three equations only. In the second case, however, if we consider the only admissible embedding, which is a pair of nullators, we obtain a fully-solvable network, with all its voltages and currents being zero. This observation shows that the difference in ranks is due to changes in the internal structure of the 2-port.

This internal structure change can be illustrated in a perhaps better way using

matroids. Figure 5 shows the the matroids  $\mathcal{G} \vee \mathcal{A}$  of the general, first and second case, respectively. Since the matroid of the second case is not graphic, we show its affine representation. After contracting 3, 4, 5 and 6, we obtain two parallel edges in the general case, the matroid  $\mathcal{M}_u$  of an ideal transformer. In the first case we can see that the rank of the contracted edges is only 3 and that after contraction we obtain  $\mathcal{M}_u$  of a nullator pair. In the second case after contraction the matroid becomes two loops which is the matroid  $\mathcal{M}_u$  of a norator pair.

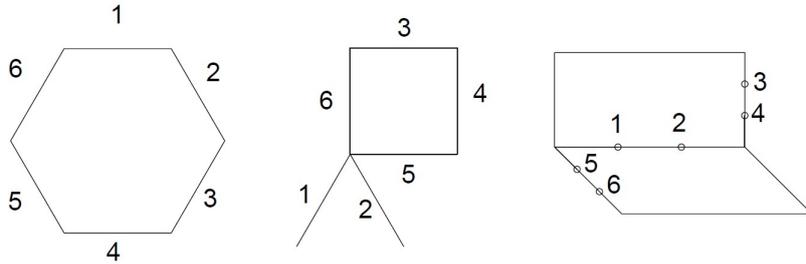


Figure 5: Matroids  $\mathcal{G} \vee \mathcal{A}$  of the general, first and second case

Both of the above cases are remarkable because they are examples that nullators and norators can be realized using reciprocal devices only. All previous realizations of these 1-ports with regular network elements used circulators to achieve this goal [10] [11]. Note, that we did not use devices that only exist in theory in either of these above cases, as an ideal transformer with the ratio  $-1$  is just an ideal transformer that has a ratio of 1 but one of its ports is flipped as shown on Figure 6.

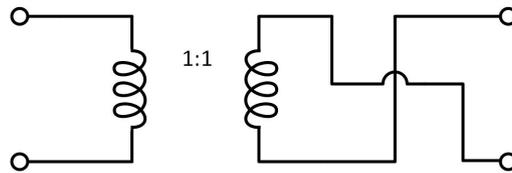


Figure 6: A transformer with  $-1$  ratio

Moreover in the second case the resulting 2-port, which is a norator pair, is neither reciprocal nor antireciprocal. This result is interesting as it was shown in [6] that these

classes are closed with respect to interconnection. In that article, however, the authors indirectly assumed that the parameters of the multiports to be interconnected are general variables, in such a way that they cannot cancel out each other. In the aforementioned cases we dropped this so called genericity assumption [12] by choosing the ratios of the transformers to be 1 and  $-1$ . This example shows how some of the most fundamental truths in electrical network theory can become false if a singularity occurs in the network.

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