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ADC testing in practice, using maximum likelihood estimation

„TDK" REPORT

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Abstract

There are several different test methods to examine analog-to-digital converters (ADCs). These techniques are available in the literature, used worldwide, and published in European and American standards. However, it is possible to extract even more information from a measurement record, than the standardized methods do.
This paper is focusing on a dynamic ADC test method, the estimation of ADC parameters with sine wave fitting. The maximum likelihood (ML) estimation of sine wave parameters is not yet standardized, but provides more accurate estimators for the sine wave on the input of the ADC, than the currently proposed least squares (LS) method. Thus the quality parameters of the device under test (the datasheet quantities) can be calculated with more precision. Naturally this theoretically more established process brings along several practical difficulties. This paper enumerates the challenges of performing ML estimation in practice, and proposes solutions for them. Also measurement and simulation examples are provided for the comparison of standardized and unstandardized test methods.

Introduction

Analog-to-digital conversion is essential when sensing of the analog environment and digital processing of signals are required. In embedded systems, the measurement and data aquisition devices necessarily involve analog-to-digital conversion. As these devices can be found in numerous products designed by electrical engineers, ADC testing became a key issue in this field. Following the development of ADC devices it is necessary develop the test methods as well. For simple, low resolution converters it is feasible to measure the transfer characteristic of the device with a control loop and a sufficiently accurate (dual slope or multislope) voltmeter [1]. For the ADCs used nowdays (resolution from 8 up to 24 bits, sampling rate up to the GS/s range) faster and more effective test methods are required. Documents like standard IEEE-1241 [1] or standard IEC 60748-4-3 [2] describe several static and dynamic methods for ADC testing. Important parameters of the device under test, like effective number of bits (ENOB), signal to noise and distortion ratio (SINAD), spurious-free dynamic range (SFDR), total harmonic distortion (THD), integral nonlinearity (INL), differential nonlinearity (DNL), and others can be calculated using these methods. However, the full information is usually not extracted from the measurement record.

Maximum likelihood (ML) parameter estimation provides consistent and asymptotically efficient estimators for parameters of the excitation signal [4]. This way dynamic ADC parameters like ENOB and SINAD can be determined more accurately. This theoretically more established test method provides unbiased datasheet quantities with the lowest possible variance.

This report is focusing on the practical realization of ML estimation for ADC testing with sinusoidal excitation signal. Chapter 1 describes the basic principles and problems of ADC testing, introducing the measurement setup. Chapter 2 presents the „state of art", the test methods that are currently described in the literature (primarily in the standards). Chapter 3 focuses on the specialization of the ML estimation for the case of ADC testing (solving the problem of the continuous parameter space and the discrete distribution of the output). These chapters do not contain new results, they are necessary to introduce the problem, and to ground the following sections containing the novel research outcome. Chapter 4 deals with the size of parameter space: describes problems and provides solutions for them. Chapter 5 is about the role of the noise in ML estimation: investigates the effect of measurement noise on the accuracy of estimators, and also describes the problems of low-noise measurements (with a proposed solution). Chapter 6 investigates the behaviour of ML estimators using simulated measurements. Results of ML and LS estimation are compared in this section. In chapter 7 implementations of these test methods are introduced in two different development enviromnents: MATLAB and LabVIEW. Chapter 8 summarizes the results and provides conclusion. An outlook to possible directions of further research is presented in chapter 9.

Chapter 1

ADC testing with sinusoidal excitation signal

ADC test methods can be classified using multiple points of view. There are techniques to examine the static behaviour and there are others to investigate the dynamic behaviour of the device under test. There are methods to evaluate the performance of the ADC in time domain and in frquency domain. The type of the excitation signal on the input is also an aspect of classification. Any type of analog signals can be used for ADC testing. Comparison of the excitation signal and the digital record always provides information about the converter. The main problem is to reconstruct the waveform of the analog signal. The input of the device under test can only be measured with an other digitizing waveform recorder. Thus to test an ADC it is requred to have an other, much better ADC. This requirement is absolutely impractical, and rises theoretical questions about how to examine the better ADCs. The way to avoid this problem is to estimate the analog signal using the digital record. This process makes the type of excitation signal important. Waveforms that can be described with only few parameters can be estimated in practice. Signals like multisine waves or arbitrary periodic signals with numerous harmonic components are barely appropriate for ADC testing, beacuse they have several signal parameters to estimate.

Sawtooth signal can be described with only three parameters (amplitude, frequency, DC component). It is also useful for histogram testing, because the probability density function (PDF) of this signal is uniform between the two extrema, and zero elsewhere. But it is impossible to generate accurate sawtooth signal because of its unlimited bandwidth, and it is difficult to examine the quality of the signal generated. Other broadband signals like triangle or square waves rise the same problem.

Exponential signals also can be used for ADC testing. These signals also can be described with three parameters, and can be generated with simple RC circuits. Naturally there are there are problems with generating pure exponential signals [5], but the quality of an exponential can be described sufficiently [6]. The framework of ADC testing with exponential stimulus is described in [7].

Choosing sine waves as excitation signal is very attractive. On the one hand with sine waves the device is tested at one given freqency, so frequency-dependent behaviour of the converter can be observed easily with multiple measurements at different frequencies. On the other hand the ipmerfections of the sine wave can be observed easily, using a spectrum analyzer. As generating pure sine wave is a very common task in electrical engineering, devices with very good parameters are available for laboratories at affordable prices. According to the subject of this report, an introduction to ADC testing with sine waves will be presented in the following sections.

1.1 Measurement setup

Examinig ADCs with sinusoidal singnals does not require complex measurement setup. A sine wave generator provides the signal that drives the analog input, and the response of the device (the digital codes) are recorded, most frequently by a PC. The generator can be a full analog circuit (based on oscillators, and analog filters) or can be an arbitrary waveform generator with ditital-to-analog converter (DAC). In each case it is essential to examine the quality of the signal created: harmonic distortion, peak-to-average ratio (PAR) and noise variance are important quantities to determine whether the generator is appropriate for the measurement or not. Recording the digital codes does not require any effort in case of standalone converters. These are usually connected to the PC with a high speed interface (USB or Ethernet), and the producer provides software support for the device. Our measurements were performed with National Instruments devices, the data were acquired with the NI LabView software. In case of ADCs integrated to a system (e.g. a DSP or a microcontroller), recording and transmitting the samples is a programming task to be solved. The block scheme of the measurement setup is on figure 1.1.

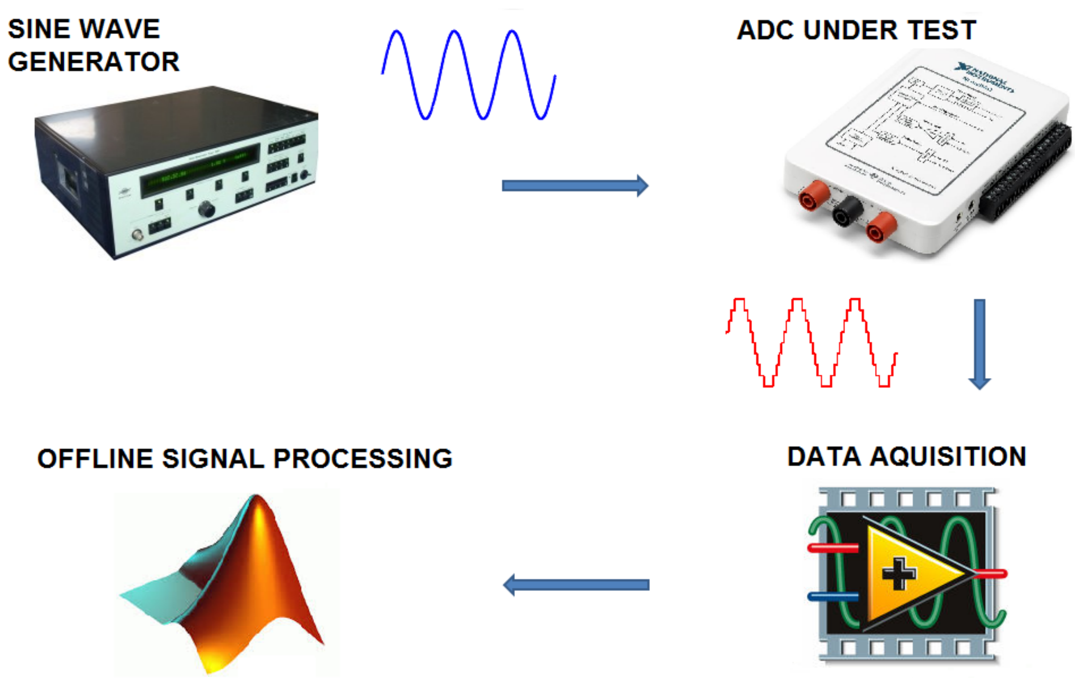

Figure 1.1: *Measurement setup for ADC testing with sine wave*

1.2 Requirements on the test environment

According to the simplicity of the measurement setup, there are only a few requirements to satisfy regarding the test envinronment. The pureness of the sine wave is a key issue in both static, both dynamic examination of the ADC. In histogram testing with sine wave the calculations are based on the PDF of the pure sine wave. Harmonic distortion changes the PDF of the real excitation signal, thus calculations with the PDF of pure sine wave can be misleading. In dynamic testing a single sine wave is fitted to the measurement record. If the record conatins higher harmonic components, these appear in the difference of the record and the fitted sine wave. This difference is treated as the noise and distortion of the device under test. If the original excitation signal is distorted, it eventuates fake test results: harmonic distortion of the device appears to be higher than its real value. Presence noise also can mislead the evaluation of the measurement. Histogram testing is surprisingly robust regarding the noise on the excitation signal: as noise changes the PDF of sine wave mostly near the extrema, effect of noise can be eliminated with high amplitude (120..150 % full scale) sine waves, that overdrive the converters [8]. Sine wave fitting methods are more sensitive to analog noise: similarly to harmonic distortion, noise in the measurement record is treated as noise of the ADC, so parameters like ENOB and SINAD seem to be worse than their actual value. On the other hand noise is an important signal parameter in ML estimation, and numerical optimization cannot be performed without the presence of some noise on the analog side. Role and properties of noise are detailed in chapter 5.

Chapter 2

Standards for ADC testing

Digital measurement of electrical quantities is essential wherever industry has taken root. Thus ADCs are manufactured and used wordlwide. However there are only two internationally recognized organizations that provide standards for ADC testing. The American IEEE (Institute of Electrical and Electronics Engineers) and the European IEC (International Engineering Consortium) both have elaborated documents that clarify the terminology and test methods. Outside Europe and America the following cases are possible:

- one of the IEEE and IEC standars is used
- a mixture of the American and European standards is used
- other public documents, like [9] are used
- terminology and test methods differ from company to company.

Japanese engineer and researcher *Hauro Kobayashi* writes „In most Japanese companies ADC standards belong to engineers and there is no comprehensive document that can be used as a Bible of ADC standards and testing methods" ([10]). This article was published in 2001, when both standards IEEE-1241, IEEE-1057 and IEC-60748-4 were available.

While the usage of these documents is still confused in several cases, there are attempts to harmonize the American and European standards [11]. Nevertheless a unified and wordwide used global standard seems to be a remote possibility only. The best choice nowdays is to choose one of the documents released, and to use it consistently. The following sections describe the main traits of the internationally used standards, deal with the advantages and disadvantages of them and counts the deficiencies of them.

2.1 American standards

The American organization of electrical and electronics engineers (IEEE) provides two documents concerning ADC testing. Standard IEEE-1057 (last revised in 2007) deals with digitizing waveform recorders, thus its scope is wider than the scope of standard IEEE-1241 (last revised in 2010) that focuses on analog-to-digital conversion only.

2.1.1 Standard IEEE-1057

As this document was developed to define specifications and to describe test methods for each device that records electrical waveforms digitally, aspects outside the problem space of analog-to-digital conversion are also mentioned in this standard. Digital oscilloscopes, digitizing waveform analyzers and recorders contain several other elemnts beyond ADCs, that are subject of examination. According to these requirements, this document provides definitions and specifications for the following fields of waveform analysis:

- **Input impedance**: the analog input of the device can be examined with DC excitation and AC excitation at various frequencies. Usage of a vector impedance meter is recommended. If measured impedances fit the model, input impedance can be expressed with the parallel combination of a capacitance and a resistance. Measurement of the reflection coefficient is also recommended using a time domain reflectometer (TDR). Input impedance can also be calculated from the value of the reflection coefficient, measured with a high quality cable, whose impedance is known accurately.

- **Crosstalk**: in case of multichannel devices crosstalk can be measured at any desired frequencies. This quantity is defined for each channel: to measure the multichannel crosstalk all *other* channels shall be driven with the maximum amplitude sine wave that can be applied. It is required to use common phase sine waves to maximize the effect of interference.

- **Common-mode rejection ratio (CMRR)**: for signal analyzers with differential input other quantities can be specified. CMRR can be measured using a large mplitude sine wave that drives both terminals of the differential input. This way the excitation signal contains no differential mode component, but a large common mode component. In the case of ideal device no signal should be recorded. The sine wave that appears in the waveform record is the effect of the common mode signal applied. Comparing the amplitude of the applied and measured sine wave, CMRR can be calculated.

- **Trigger delay and trigger jitter**: In digitizing oscilloscopes time delay between the trigger pulse, and the first sample recorded is also an important parameter. Trigger jitter is the deviation of the trigger delay. It can be estimated performing several trigger delay measurements. (Affined quantities aperture delay and aperture jitter are properties of ADCs, so these are detailed in standard IEEE-1241)

- **Out-of-range recovery**: input voltages outside the full scale range (FSR) of the digitizer can cause stray effects like saturation of an amplifier, or thermal effects bacause of the higher power dissipation. The recovery time of the device under test from the anomalous state to the state of normal operation can be measured using sine wave fitting. A large ($A < 120\% FSR$) and slow ($f < 0.05 f_{sampling}$) sine wave shall be recorded. When the signal leaves the treshold domain and returns to the full scale range, abnormal samples (samples not fitting to the sine wave) can be observed.

Time delay between the return of the signal and the disappearance of the abnormal samples can be estimated. The so called „mod T plot" of residuals can be very useful to observe out-of-range recovery.

In the paragraphs above fundamental parameters, measurement setups and evaluation procedures were itemized regarding digital waveform measurement and analysis *excluding* analog-to-digital conversion. As analog-to-digital conversion is a key issue in digital waveform measurement, several chapters regarding ADCs appear in this standard. Thus there is a large overlap between the standards IEEE-1057 and IEEE-1241. Important quality parameters and standardized test methods for ADCs are detailed in section 2.1.2.

2.1.2 Standard IEEE-1241

This document is focusing on terminology and test methods for analog-to-digital converters only. Overlapping parts of standard IEEE-1057 and standard IEEE-1241 are introduced in this section. The main quality parameters of an ADC are the followings:

- **Gain and offset**: The spread of the *real* full scale range can be expressed with the highest and lowest code transition voltage ($T[1]$ and $T[2^N - 1]$). However, it is more expressive to define quantities *offset* and *gain*. Offset denotes the difference between the endpoints of the theoretical and real full scale range, gain denotes the ratio of the theoretical and the real average code bin width.

- **Linearity**: The static transfer characteristic of and ADC is the set of code transition levels. Code transition level $T[k]$ is a DC voltage value, where the probability of output code $k-1$ and k are equal (50-50%). In case of linear ADCs distances between transition levels are equal, thus:

$$T_{ideal}[k] = T[1] + Q \cdot (k - 1) \tag{2.1}$$

where Q denotes the average code bin width. *Integral nonlinearity* (INL) is the difference between the ideal and real code transition levels. Thus INL can be evaluated for each code transition:

$$\mathrm{INL}[k] = \frac{T[k] - T_{ideal}[k]}{Q} \tag{2.2}$$

On datasheets integral nonlinearity appears in two different ways. In some cases a typical INL characteristic of the device is provided, INL[k] is plotted according to the transition levels. In other cases INL is given as the maximal absolute linearity error, thus

$$\mathrm{INL} = \max_{k=1}^{2^N-1}(|\mathrm{INL}[k]|) \tag{2.3}$$

- **Noise and distortion**: The most important quality parameters of an ADC are related to noise and distortion. Signal to noise ratio (SNR), signal to noise and distortion ratio (SINAD), effective number of bits (ENOB), are both evaluated using the sine wave fit method in time domain. In these measurements a pure analog sine wave is recorded, then samples of a sine wave are fitted to the digital record. Sine wave fitting algorithms are detailed in section 2.1.3.

- **Step response**: Examining the step response is a very manifest test method for any device in signal processing. For ADC testing it is required that the imperfections of the test signal (transition duration, settling time and overshoot) shall not be greater than the 25 % of the imperfections expected from the device under test. By processing the record of the step response, quantities of the ADC can be calculated such as transition duration, overshoot, settling time, limits of slew rate, and many other parameters.

- **Frequency response parameters**: Examining the performance of the converter in the frequency domain is also important to provide useful information for the users. Frequency domain is usually sweeped from DC to the *Nyquist frequency* of the device. To investigate the magnitude of aliasing it also makes sense to use excitation signals over the Nyquist frequency. Bandwidth and gain flatness can be examined observing the amplitude response of the ADC in various frequencies. Most of the converters have no lower -3dB frequencies, as they are capable to measure DC voltages. The higher boundary of the passband is more important and can be expalined with multiple reasons. The limited bandwidth of the analog anti-aliasing filter (AAF) reduces the bandwidth of the entire device. Also slew rate limitations can be sources of lower amplitude response in higher frequencies. Gain flatness is a useful quantity do describe the wost case gain error in the passband:

$$E_G(f) = \frac{G(f) - G(f_{ref})}{G(f_{ref})} \tag{2.4}$$

$$E_G = \max_{i=1}^{N} |E_G(f_i)| \tag{2.5}$$

  where $f_r ef$ is the reference frequency, where the amplitude response is considered ideal (in DC coupled devices usually the zero frequency), and f_i denotes the freqency values where the gain was examined.

2.1.3 Sine wave fitting in least squares sense

Three parameter sine wave fit in least squares sense

Fitting samples of a sine wave to a measurement record can be performed minimizing the following cost function:

$$\mathrm{CF}_{LS} = \sum_{m=1}^{M} (y[m] - A\cos(2\pi f t_m) - B\sin(2\pi f t_m) - C)^2 \tag{2.6}$$

where M denotes the length of the record, t_m is the sampling time of the m^{th} sample, A is the cosine coefficient, B is the sine coefficient, and C is the DC component, and f is the frequency of the of the fitted sine wave. This cost function shall be minimized with respect to three parameters, A, B and C. The frequency of the signal is fixed. This least squares problem is linear in parameters, solution can be calculated in one step with simple matrix operations:

$$\mathbf{p} = (\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}\mathbf{y} \tag{2.7}$$

where

$$\mathbf{p} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \tag{2.8}$$

and

$$\mathbf{D} = \begin{bmatrix} \cos(2\pi f t_1) & \sin(2\pi f t_1) & 1 \\ \cos(2\pi f t_2) & \sin(2\pi f t_2) & 1 \\ \vdots & \vdots & \vdots \\ \cos(2\pi f t_M) & \sin(2\pi f t_M) & 1 \end{bmatrix} \tag{2.9}$$

and $\mathbf{y}$ is the measurement record:

$$\mathbf{y} = \begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[M] \end{bmatrix} \tag{2.10}$$

Four parameter sine wave fit in least squares sense

The minimization of the cost function is more complicated, when the frequency of the signal is also a parameter to estimate. In this case an iterative solution can be performed: an initial frequency estimator f_0 shall be computed, and initial estimators for A_0, B_0 and C_0 shall be determined using three-parameter LS fit with given frequency f_0. Then four parameters A_i, B_i, C_i and Δf_i are to be calculated with the following equation:

$$p_i = \begin{bmatrix} A_i \\ B_i \\ C_i \\ \Delta f_i \end{bmatrix} = (\mathbf{D_i}^T\mathbf{D_i})^{-1}\mathbf{D_i}\mathbf{y} \tag{2.11}$$

where

$$
\mathbf{D_i} = \begin{bmatrix} \cos(2\pi f_i t_1) & \sin(2\pi f_i t_1) & 1 & -A_i t_1 \sin(2\pi f_i t_1) + B_i t_1 \cos(2\pi f_i t_1) \\ \cos(2\pi f_i t_2) & \sin(2\pi f_i t_2) & 1 & -A_i t_2 \sin(2\pi f_i t_2) + B_i t_2 \cos(2\pi f_i t_2) \\ \vdots & \vdots & \vdots & \vdots \\ \cos(2\pi f_i t_M) & \sin(2\pi f_i t_M) & 1 & -A_i t_M \sin(2\pi f_i t_M) + B_i t_M \cos(2\pi f_i t_M) \end{bmatrix} \quad (2.12)
$$

Then frequency shall be adjusted: $f_i = f_{i-1} + \Delta f_{i-1}$, and matrix $\mathbf{D_i}$ shall be calculated again with f_i. Iteration can be terminated when frequency step Δf reaches a boundary defined previously.

2.2 European standards

In Europe, standards for electronic devices are provided by the International Electrotechnical Comission (IEC). Standard IEC-60748-4-3 [2] is regarding the dynamic criteria for analog-to-digital converters. This document has been elaborated by the subcommittee 47A (integrated circuits) of IEC technical committee 47 (semiconductor devices). First edition of the standard has been released in 2006.

2.2.1 Standard IEC-60748-4-3

This document is a brief summary of terminology and test methods for ADC testing. Contains only 36 pages instead of the american standards, that contain approximately 150 pages. However, parts of the standard IEC-60748-4 „Semiconductor devices - integrated circuits" [3] are referenced in this document. Terms and definitions for static behaviour and characteristics of ADCs are described in Chapter II and Chapter III of the document referred. In the fist part of standard IEC-60748-4-3 fundamental terms like coherency, equivalent sampling, code transition, linearity, SINAD, ENOB and others are defined very briefly. In the second part dynamic test methods like sinusoidal excitation, step response examination, ramp signal excitation are detailed. In these sections definitions for calculated quantities such as THD or linearity error are provided. Mathematical derivations, restrictions on signal generators and figures regarding the measurement setups appear in the annexes.

Chapter 3

Theoretical background of maximum likelihood (ML) estimation for ADC testing

3.1 General principles of ML estimation

Maximum likelihood estimation is a very attractive method to estimate parameters of a model observing the output of the system. To perform estimation, a set of independent observations $(x_1, x_2, ..., x_n)$ are required. The parameters of the model (α) can be estimated via the so called *likelihood function*. To specify the likelihood function, it is necessary to express the *joint density function* of all observations:

$$f(x_1, x_2, ..., x_n|\alpha) = \prod_{i=1}^{n} f(x_i|\alpha) \tag{3.1}$$

where $f(x_i|\alpha)$ denotes the conditional probability density function of the observations with parameter α. The advantage of the independency of observations is obviously taken in this case. As we have fixed observations and an unknown parameter vector to estimate, the joint density function can be used in the following way:

$$L(\alpha|x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n|\alpha) = \prod_{i=1}^{n} f(x_i|\alpha) \tag{3.2}$$

The likelihood function (denoted by L) specifies the „agreement" of the parameter vector with the observations. Maximum likelihood estimator of α can be calculated maximizing the likelihood function:

$$\hat{\alpha}_{\mathrm{ML}} = \arg\max_{\alpha} L(\alpha|x_1, x_2, ..., x_n) \tag{3.3}$$

To ease numerical computation, usually *log-likelihood* or *average log-likelihood* functions are optimized with respect to the parameters. Log-likelihood function is the natural logarithm of the likelihood function:

$$l = \ln L(\alpha|x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \ln f(x_i|\alpha) \tag{3.4}$$

Average log-likelihood is normalized with the number of observations:

$$l_{\text{avg}} = \frac{1}{n} \ln L = \frac{1}{n} \sum_{i=1}^{n} \ln f(x_i|\alpha) \tag{3.5}$$

Since logarithm function is strictly increasing, extrema of likelihood and log-likelihood functions are identical. However a sum can be derived more easily than a product, thus for optimization algorithms that require to calculate gradient or Hesse-matrix, it is strongly recommended to use the log-likelihood instead of the likelihood function.
Maximum likelihood estimator has the following properties [4]:

- **Consistency**: with sufficiently large number of observations, the *real* model parameters can be approximated with arbitrary precision. The ML estimator convergs to the real value of the parameters.

$$\lim_{M\to\infty} \mathrm{P}[|\hat{\alpha}_{\mathrm{ML}} - \alpha| > \epsilon] = 0 \tag{3.6}$$

  where M is the number of independent observations, and ϵ is an arbitrary positive real number.

- **Efficiency**: with number of observations tending to infinity, variance of the estimators reaches the Cramér-Rao bound.

- **Asymptotic normality**: the distribution of ML estimators is Gaussian with α mean, when number of samples tends to infinity.

Cramér-Rao bound, Fisher information, and covariance of estimators in case of ML estimation for ADC testing will be discussed in section 3.2.

3.2 ML estimation specialized for ADC testing

For maximum likelihood estimation of ADC testing the following model has been developed [12]. The converter is described with a set of code transition levels. Transition level $T[k]$ is the value of the input voltage, that results code $k-1$ with probaility of 50%, and code k with probability of 50% as well. The N-bit quantizer provides codes from 0 up to 2^N-1, and has 2^N-1 transition levels. The reduced full scale of the converter is the voltage range between $T[1]$ and $T[2^N-1]$. Voltage values above the highest transition level result code 2^N-1 and voltages below the lowest transition level result code 0. Quantization can be described with a function $q(x)$ where

$$q(x) = \begin{cases} 0, & \text{if } x < T[1] \\ m, & \text{if } T[m] < x < T[m+1] \\ 2^N - 1, & \text{if } x > T[2^N - 1] \end{cases} \tag{3.7}$$

The sinusoidal excitation signal can be described with four parameters:

$$x(t) = A\cos(2\pi ft) + B\sin(2\pi ft) + C \tag{3.8}$$

where A is the cosine coefficient, B is the sine coefficient, and C denotes the DC component of the signal. The frequency of the sine wave is denoted with f. The electronic noise of the devices, and the imperfections of the measurement environment are modelled with additional noise on the excitation signal. Multiple noise models can be suitable for ML estimation, these will be itemized in chapter 5. The most manifest is to assume Gaussian noise with zero mean and σ standard deviation. Let $n(t)$ denote the realization of the additive noise. In this model the spectrum of the noise is white, so $n(\tau_1)$ and $n(\tau_2)$ are independent, if $\tau_1 \neq \tau_2$.
This noisy sine wave is quantized and sampled (the sequence is interchangeable), thus the output of the ADC can be described this way:

$$y(k) = q(x(t_k) + n(t_k)) \tag{3.9}$$

where t_k denotes the k_{th} sampling time moment (k = 1..M).
The parameters of the model to be estimated to be estimated are the followings:

- The code transition levels of the quantizer: $T[1], T[2], ..., T[2^N - 1]$
- The cosine coefficient of the sine wave: A
- The sine coefficient of the sine wave: B
- The DC component of the sine wave: C
- The frequency of the sine wave: f
- The standard deviation of noise on the excitation signal: σ

As uniform sampling is assumed (effects of incidental nonideal sampling are not considered in this model), the frequency of the sine wave can be described using the angular frequency normailzed to the sampling frequency:

$$\theta = \omega T_s = 2\pi \frac{f}{f_s} \tag{3.10}$$

where T_s is the sampling time, and f_s denotes the sampling frequency. Thus the parameter vector to be estimated is the following:

$$\mathbf{p} = \begin{bmatrix} A \\ B \\ C \\ \theta \\ \sigma \\ T[1] \\ T[2] \\ \vdots \\ T[2^N-2] \\ T[2^N-1] \end{bmatrix} \tag{3.11}$$

To express the likelihood of the parameters, it is necessary introduce a vector of discrete random variables, denoted by $\mathbf{Y}$. Value $Y(k)$ belongs to the k^{th} sample of the measurement record and can achieve 2^N values: it can be any of the output codes of the ADC form 0 to 2^N-1 with a given probaility. These probailities can be described using the error function:

$$\text{erf}(x) = \frac{2}{\pi}\int_0^x e^{-z}\text{d}z \tag{3.12}$$

$$\text{P}(Y(k)=0) = \frac{1}{2}\left[\text{erf}\left(\frac{T[1]-x(t_k)}{\sigma\sqrt{(2)}}\right)+1\right] \tag{3.13}$$

$$\text{P}(Y(k)=2^N-1) = \frac{1}{2}\left[1-\text{erf}\left(\frac{T[2^N-1]-x(t_k)}{\sigma\sqrt{(2)}}\right)\right] \tag{3.14}$$

$$\text{P}(Y(k)=l) = \frac{1}{2}\left[\text{erf}\left(\frac{T[l+1]-x(t_k)}{\sigma\sqrt{(2)}}\right)-\text{erf}\left(\frac{T[l]-x(t_k)}{\sigma\sqrt{(2)}}\right)\right] \tag{3.15}$$

where l $=$ 1..2^N-2

To avoid using three different cases it is useful to define two „virtual" transition levels of the ADC: $T[0] = -\infty$ and $T[2^N] = +\infty$. This way the value of $Y(k)$ can be expressed in one equation:

$$\text{P}(Y(k)=l) = \frac{1}{2}\left[\text{erf}\left(\frac{T[l+1]-x(t_k)}{\sigma\sqrt{(2)}}\right)-\text{erf}\left(\frac{T[l]-x(t_k)}{\sigma\sqrt{(2)}}\right)\right] \tag{3.16}$$

where l $=$ 0..2^N-1

The likelihood function for the entire measurement is:

$$L(\mathbf{p}) = \prod_{k=1}^{M}\text{P}(Y(k)=y(k)) \tag{3.17}$$

where $y(k)$ is the k^{th} sample of the digital record. Merging the equations above, one can express the likelihood function this way:

$$L(\mathbf{p}) = \prod_{k=1}^{M} \frac{1}{2} \left[\operatorname{erf} \left(\frac{T[y(k)+1] - x(t_k)}{\sigma \sqrt(2)} \right) - \operatorname{erf} \left(\frac{T[y(k)] - x(t_k)}{\sigma \sqrt(2)} \right) \right] \tag{3.18}$$

For computations it is feasible to define a cost function, which is the negative log-likelihood function:

$$\mathrm{CF}(\mathbf{p}) = -\ln L(\mathbf{p}) \tag{3.19}$$

Chapter 4

Problems and solutions concerning the size of the parameter space

As it is described in chapter 3, the number of parameters strongly depends on the bit number of the conversion. As the parameter vector contains five signal parameters (A,B,C,θ,σ) and $2^N - 1$ code transition levels, the length of parameter vector rises exponentially depending on the number of bits. As quantizers used in practice usually provide from 8 upto 24 bits resolution, the explosion of the parameter space rises serious challenges regarding computation time and numerical stability. To perform even the most simple numerical optimization algorithm, the negative gradient method, it is required to calculate the gradient of the cost function in each iteration:

$$\nabla \mathrm{CF}(\mathbf{p}) = \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial \mathbf{p}} = \begin{bmatrix} \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial A} \\ \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial B} \\ \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial C} \\ \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial \theta} \\ \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial \sigma} \\ \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial T[1]} \\ \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial T[2]} \\ \vdots \\ \frac{\partial \mathrm{CF}(\mathbf{p})}{\partial T[2^N-1]} \end{bmatrix} \tag{4.1}$$

The most obvious problem is to calculate $2^N + 4$ partial derivatives in each iteration cycle. However, this challange can be answered allocating suffucient brute force (e. g. GPU-s) for the computation. The properties of partial derivatives with respect to the transition levels are more problematic. To examine this question, it is necessary to express the partial derivatives. The cost function is the negative log-lokelihood function:

$$\mathrm{CF}(\mathbf{p}) = -\sum_{k=1}^{M} \ln \frac{1}{2} \left[\mathrm{erf}\left(\frac{T[y(k)+1] - x(k)}{\sqrt{2}\sigma} \right) - \mathrm{erf}\left(\frac{T[y(k) - x(k)]}{\sqrt{2}\sigma} \right) \right] \tag{4.2}$$

where $x(k)$ is the k^{th} sample of the pure sine wave:

$$x(k) = A\cos(k\theta) + B\sin(k\theta) + C \tag{4.3}$$

To ease overview of formulas, it is feasible to introduce the following notation:

$$\arg(k) = \operatorname{erf}\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{T[y(k)-x(k)]}{\sqrt{2}\sigma}\right) \tag{4.4}$$

The first order partial derivative of the cost function are expressed below:

$$\frac{\partial \mathrm{CF}}{\partial A} = -\sum_{k=1}^{M} \frac{1}{\arg(k)} \frac{2}{\pi} \left(e^{\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{\cos(k\theta)}{\sqrt{2}\sigma} - e^{\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{\cos(k\theta)}{\sqrt{2}\sigma} \right) \tag{4.5}$$

$$\frac{\partial \mathrm{CF}}{\partial B} = -\sum_{k=1}^{M} \frac{1}{\arg(k)} \frac{2}{\pi} \left(e^{\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{\sin(k\theta)}{\sqrt{2}\sigma} - e^{\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{\sin(k\theta)}{\sqrt{2}\sigma} \right) \tag{4.6}$$

$$\frac{\partial \mathrm{CF}}{\partial C} = -\sum_{k=1}^{M} \frac{1}{\arg(k)} \frac{2}{\pi} \left(e^{\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{1}{\sqrt{2}\sigma} - e^{\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{1}{\sqrt{2}\sigma} \right) \tag{4.7}$$

$$\frac{\partial \mathrm{CF}}{\partial \theta} = -\sum_{k=1}^{M} \frac{1}{\arg(k)} \cdot \frac{\partial \arg(k)}{\partial \theta} \tag{4.8}$$

where

$$\frac{\partial \arg(k)}{\partial \theta} = \frac{2}{\pi} \left(e^{\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{A\sin(k\theta)k - B\cos(k\theta)k}{\sqrt{2}\sigma} - e^{\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{A\sin(k\theta)k - B\cos(k\theta)k}{\sqrt{2}\sigma} \right) \tag{4.9}$$

$$\frac{\partial \mathrm{CF}}{\partial \sigma} = -\sum_{k=1}^{M} \frac{1}{\arg(k)} \cdot \frac{\partial \arg(k)}{\partial \sigma} \tag{4.10}$$

where

$$\frac{\partial \arg(k)}{\partial \sigma} = \frac{2}{\pi} \left(e^{\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{T[y(k)]-x(k)}{\sqrt{2}\sigma^2} - e^{\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma^2} \right) \tag{4.11}$$

The partial derivatives with respect to transition levels are more complicated:

$$\frac{\partial \mathrm{CF}}{\partial T[l]} = -\sum_{k=1}^{M} \frac{1}{\arg(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \tag{4.12}$$

where

$$\frac{\partial \arg(k)}{\partial T[l]} = 0 \tag{4.13}$$

if $y(k) \neq l-1$ and $y(k) \neq l$. For the majority of k values (for the majority of the samples in the record), the elements of the sum are zero. Elements are nonzero only if $y(k) = l-1$, in this case

$$\frac{\partial \arg(k)}{\partial T[l]} = \frac{2}{\pi} e^{\left(\frac{T[l]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{1}{\sqrt{2}\sigma} \tag{4.14}$$

and if $y(k) = l$, in this case

$$\frac{\partial \arg(k)}{\partial T[l]} = -\frac{2}{\pi} e^{\left(\frac{T[l]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{1}{\sqrt{2}\sigma} \tag{4.15}$$

Summarizing the facts detailed above, optimizing the cost function in the entire parameter space is very problematic. The number of parameters increases exponentially depending on the bit number of the device under test, and partial derivatives with respect to the transition levels are very small, or even can be identically zero. However, it is possible to find a satisfying approximate solution for the maximum likelihood problem.

4.1 Reduction of the parameter space

To perform approximation, it is required to get transition level estimators using other methods, and optimizing the cost function with respect to the parameters that are dominantly determine the likelihood: the four signal parameters (A, B, C, θ), and the deviation of the noise (σ). Noise is a special parameter in ML estimation, properties and effects of noise are detailed in chapter 5. To estimate static transfer characteristic of the ADC it is manifest to use histogram testing. As probability density function (PDF) of a sine wave is well-known, it is possible to estimate intergal nonlinearity (INL) and differential nonlinearity (DNL) of the device, using a sinusoidal excitation signal. It is very important to be able to perform histogram test using a sine wave. As there is no more information available, than the measurement record, histogram testing and sine wave fitting shall be computed using the same data: the samples of the sine wave recorded.

4.2 Appropriate estimation of transition levels

To perform histogram test properly, there are some requirements that are easy to describe in theory, and can barely be satified in practice. These requirements are to ensure that the distibution of samples in the phase space is uniform between 0 and 2π.

- **Coherence**: a record is cohenent, if integer number of sine wave periods are sampled. For M samples and T_s sampling time

$$M \cdot T_s = N \cdot T_0 \tag{4.16}$$

where N is a positive integer, and T_0 is the reciprocal of the sine wave frequency.

- **Relatively prime condition**: the number of periods in a record (N) and the number of samples (M) shall be relatively primes. This condition ensures that each sample is in a uniqe phase position.

To fulfil these requirements, it is necessary to set the frequency of the sine wave generator appropriately. However, as both the ADC under test and the generator have frequency uncertainties, the record can be misleading. It is necessary to investigate coherency *after* the measurement, using a very accurate frequency estimator. This way, based on an estimator of the *real* f/f_s value, it is possible to discard samples of the record outside the coherent part, or to suggest to set other frequency values on the generator, to achieve better phase distribution of samples. [13] provides detailed information about the framework of coherency examination. Using proper measurement records, the transition levels can be estimated with any desired accuracy at a given confidence level using enough number of samples in a record. Formulas to calculate the required number of samples are described in [8]. As required record length can be long at higher bit numbers (e. g. for a 16-bit converter it is not superfluous to acquire multiple million samples), it is a feasible option to estimate the transition levels from the entire record, and to perform the ML sine wave fit using a shorter section. This procedure might be useful to speed up computation. Behaviour of estimators with increasing amount of data is examined in chapter 6.

Chapter 5

The role of noise in ML parameter estimation

5.1 Noise models

There are two requirements regarding the noise model used for ML estimation in ADC testing. On the one hand this model must be compatible with the physical phenomena that appear in electronic devices, on the other hand it must be treatable in the mathematical model.

5.1.1 Gaussian noise model

In this model the noise is assumed to be Gaussian with $\mu = 0$ mean and σ standard deviation. Thus the probability density function (PDF) of the noise is

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{5.1}$$

Assuming that the electronic noise has many independent sources, and the resultant noise is the sum of many independent random variables, the Gaussian noise model is very attractive. Mathematical properties of this distribution are also suitable: the PDF can be derived any times anywhere, thus calculating partial derivatives of the cost function with respect to the parameters can be performed. To validate this model, real PDF of noise can be estimated taking long measurements with zero excitation, and creating a histogram. These results show that Gaussian model is a good approximation, however real measurement noise significantly differs from exact Gaussian distribution. Figure 5.1 shows a histogram of 1 million measured samples of noise.

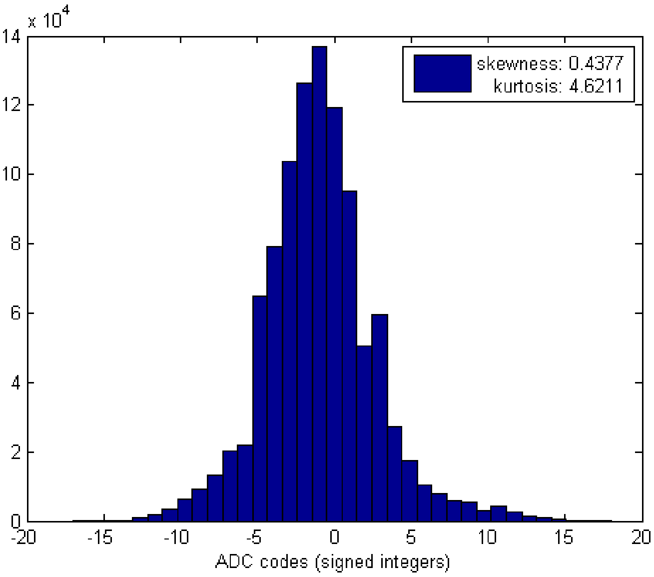

Figure 5.1: *Histogram of noise using 1 million samples*

The distribution is not symmetric (skewness is nearly 0.5), and the kurtosis is more than 150% of the kurtosis of normal distribution. To model the more outlier-prone distribution of noise, other noise models can be considered.

5.1.2 Laplace noise model

The noise can also be modeled with a random variable following the Laplace-distribution with $\mu = 0$ mean and λ scale parameter. Thus the PDF of the noise is

$$f(x) = \frac{1}{2}\lambda e^{\lambda|x-\mu|} \tag{5.2}$$

This distribution assumes more outliers, and has an other desirable numerical property: the cumulative distribution function (CDF) can be evaluated using simple exponential calculations:

$$F(x) = \int_{-\infty}^{x} f(u)\mathrm{d}u = \begin{cases} \frac{1}{2}e^{\lambda(x-\mu)}, & \text{if } x < \mu \\ 1 - \frac{1}{2}e^{\lambda(x-\mu)}, & \text{if } x \geq \mu \end{cases} \tag{5.3}$$

This fact can be important regarding computation time: using Gaussian noise model the the error function is evaluated very numerous times (at least two times for each sample of the record). Evaluating a simple exponential is significantly faster than evaluating the error function that only can be approximated using numerical methods.

The derivatives of the PDF exist anywhere except for the 0. Thus the cost function can be derived partially with respect to the parameters except for those situations, when a sample of the pure sine wave in the model is equal with the value of a transition level. In these cases PDF cannot be derived, thus partial derivation of cost function cannot be performed.

5.1.3 Spectral distribution of noise

Maximum likelihood estimation requires independent observations on the output of a system to be estimated. To fulfil this requirement, samples of additive noise shall be independent at any sampling frequency. Thus spectral distribution of noise must be uniform. Examining long measurement records of noise show that white noise model is a very good approximation of the real noise spectrum. Figure 5.2 displays the amplitude spectrum of a noise measurement record containing 2 million samples. The sampling frequency is $f_s = 200$ kHz, thus the frequency resolution is $\Delta f = 0.1$ Hz. Some minor peaks appear near 20 kHz: these indicate the electromagnetic pollution of switching-mode power supplies. The emission of powerlines can also be detected at 50 Hz, however these tiny imperfections no not question the validity of the white noise model.

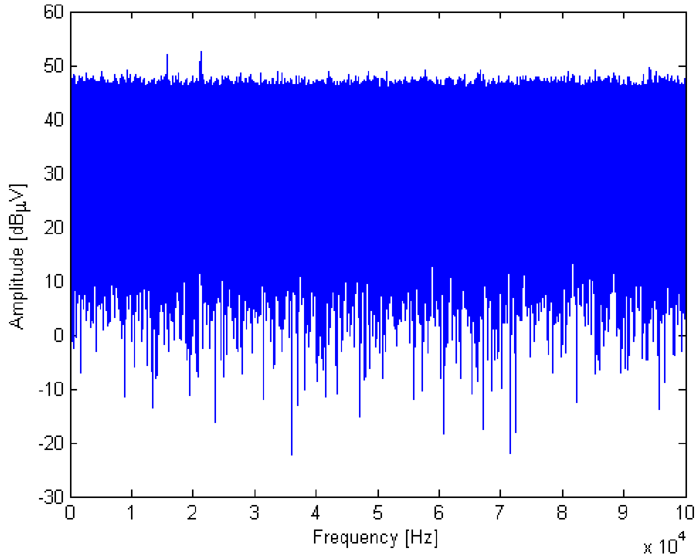

Figure 5.2: *Spectrum of measurement noise*

5.2 Using noise to relax the optimization

As it was mentioned above, noise is a special parameter of the likelihood function. On the one hand σ is one of the parameters to be estimated: the $\hat{\sigma}_{\mathrm{ML}}$ estimator provides the maximum likelihood estimation for the deviation of noise. On the other hand in case of low-noise measurements, σ is rather a tool for the optimization algorithm to find the extrema of the cost function numerically. This special behaviour is detailed below.

As $L(\mathbf{p})$ is a product of probabilities, defined by the error function, the value of each probability determines the overall likelihood of the measurement. For the k_{th} sample $y(k)$, the probability of being between the two corresponding transition levels $T[y(k)]$ and $T[y(k)+1]$ is the integral of the PDF of Gaussian distribution between $T[(k)]$ and $T[y(k) + 1]$. The mean of this distribution is the k^{th} sample of the pure sine wave, desribed using parameters A, B, C and θ.

$$\mathrm{P}[T[y(k)] < x(k)+n(k) \le T[y(k)+1]] = \frac{1}{2}\left[\mathrm{erf}\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)\right] \tag{5.4}$$

where $x(k)$ denotes the k^{th} sample of the pure sine wave: $x(k) = A\cos(k\theta) + B\sin(k\theta) + C$, and $n(k)$ is the k^{th} sample of the gaussian noise. With the same sine wave parameters and different noise variance these probabilities can be very different. On figure 5.3 the sample of the pure sine wave is between transition levels $T[y(k)]$ and $T[y(k)+1]$. In case of low noise variance, only output code $y(k)$ is compatible with the sine wave parameters A, B, C and θ. Increasing parameter σ, the adjacent codes also become compatible, they can occur with a finite probability.

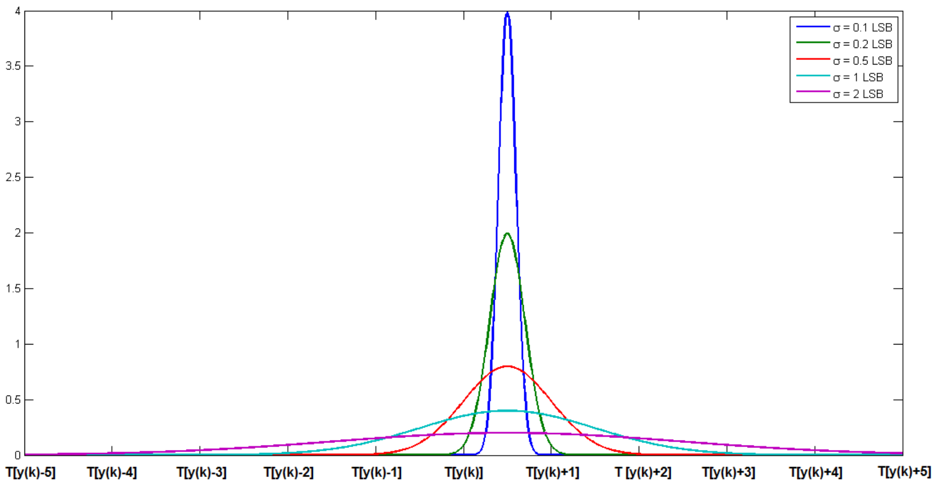

Figure 5.3: *Probability density of a noisy sample with different noise deviations*

This connection between noise and output codes can be reversed to likelihood: with a given mesurement record, the set of compatible parameter vectors depends on the noise deviation. Permitting high σ a wide range of parameter vectors become compatible with the measurement, but none of them is very likely. In case of low noise, only a narrow scale of parameter vectros is compatibe, but their likelihood is high, can be nearly 1. When noise is arbitrary low ($\lim \sigma \to 0$), each sample can be totally compatibe with the sine wave parameters (the likelihood of the parameters for that sample is 1) or can be totally incompatible (the likelihood of parameters for that sample is 0). This way the likelihood function can also be either 0 or 1 for a given parameter vector. In this case the first and second order partial derivatives (elements of the gradient and the Hess-matrix) are 0 or do not extist. Thus numerical optimization algorithms based on derivatives such as Gauss-Newton method or Levenberg-Marquardt method cannot optimize this likelihood function. The solution for this problem can be described in a few steps:

1. Calculate initial estimators for parameters A, B, C, and θ using four parameter sine wave fit in least squares sense.

2. Calculate estimators for transition levels using histogram of the record (be sure to fulfil requirements of histogram test).

3. Calculate initial estimator for noise using the quantized pure sinewave and the real measurement record:

$$\sigma_0 = \sqrt{\frac{1}{M-1}\sum_{k=1}^{M}\left(q(x_{\mathrm{LS}}(k)) - y(k)\right)^2} \tag{5.5}$$

   where $x_{\mathrm{LS}}(k)$ is the k^{th} sample of sine wave assembled using LS estimators. If $\sigma_0 = 0$ (the quantized pure sine wave is identical with the measured sine wave), or too low, increase σ_0 artificially. Setting $\sigma_0 = 0.5$ LSB is a good decision according to our experience. Depending on the numerical evaluation of error function, it is recommended to increase σ_0 artificially if σ_0 is below 0.1 LSB.

4. Start optimization. The Levenberg-Marquardt method with scale factor (λ) adjusted appripriately at each iteration converges to the minimum of the cost function. If optimization fails according to numeric problems, the initial noise estimator might be too low. It shall be increased artificially as described in step 3.

The following figures trace the progress of optimization in case of low-noise measurement. The likelihood function is displayed with respect to two parameters: A and B. DC component and frequency are kept constant, and the deviation of noise changes in each step.

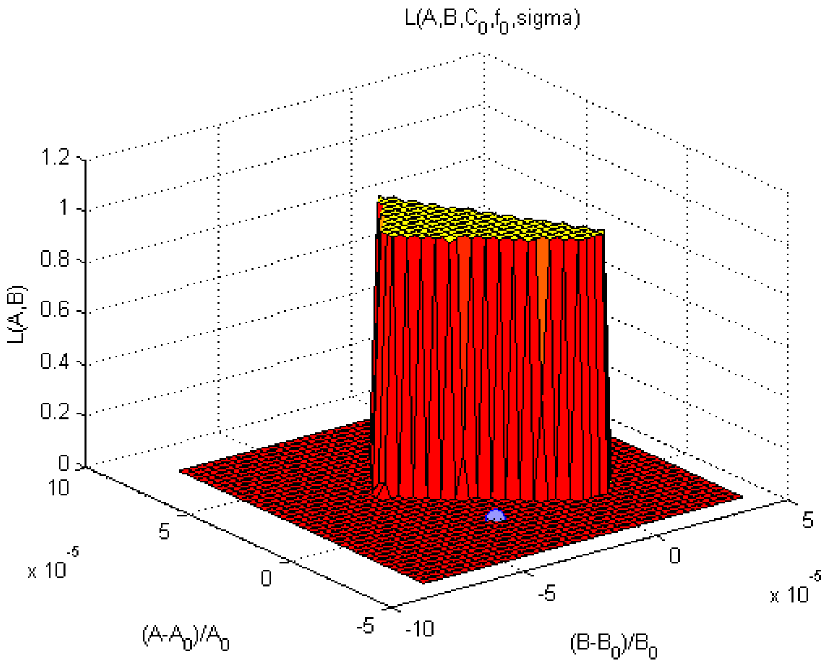

Figure 5.4: *Initial LS estimators are incompatible with the measurement: the likelihood is 0 and no derivatives can be calculated*

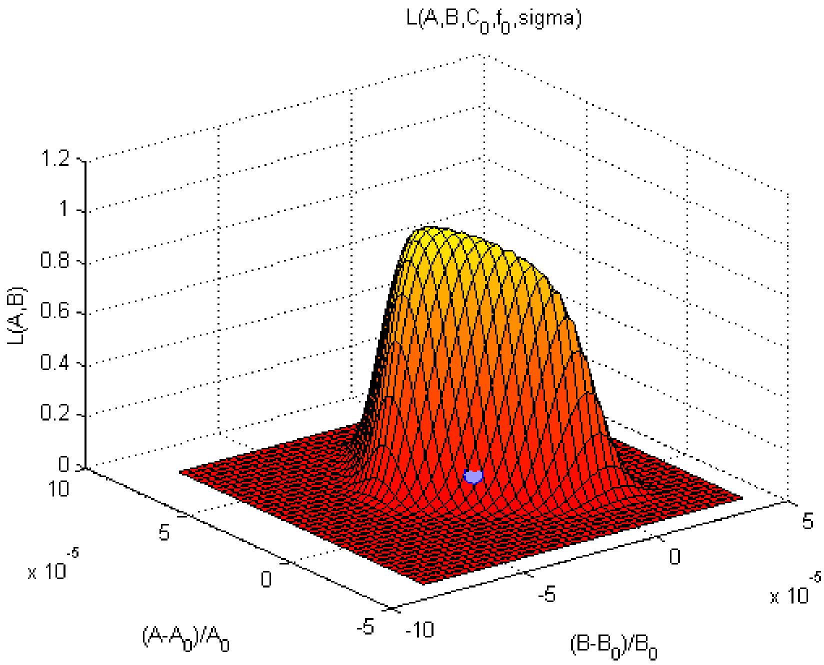

Figure 5.5: *Increasing σ smoothes the likelihood function: partial derivatives can be calculated, optimization can be initialized.*

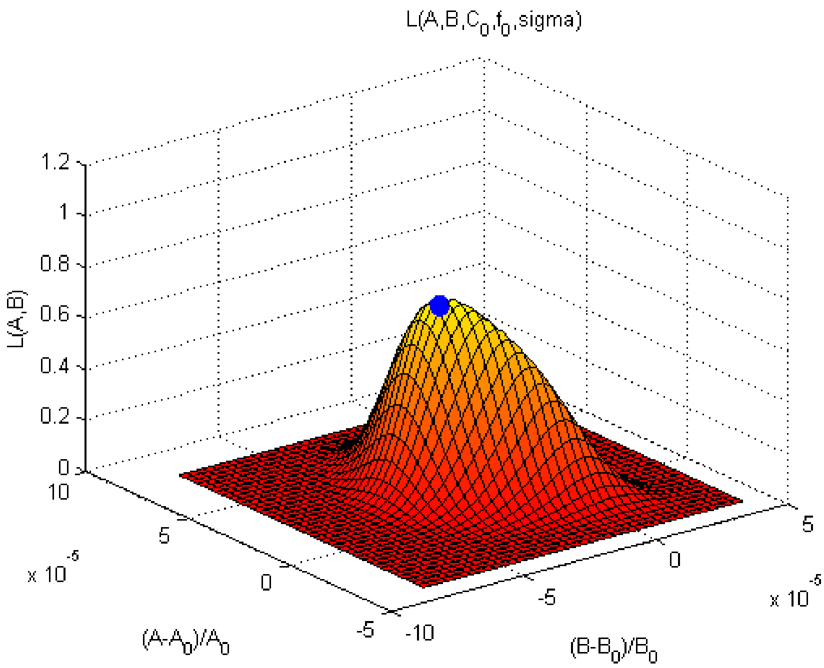

Figure 5.6: *The smoothed cost function can be optimized numerically: estimators find the extremum.*

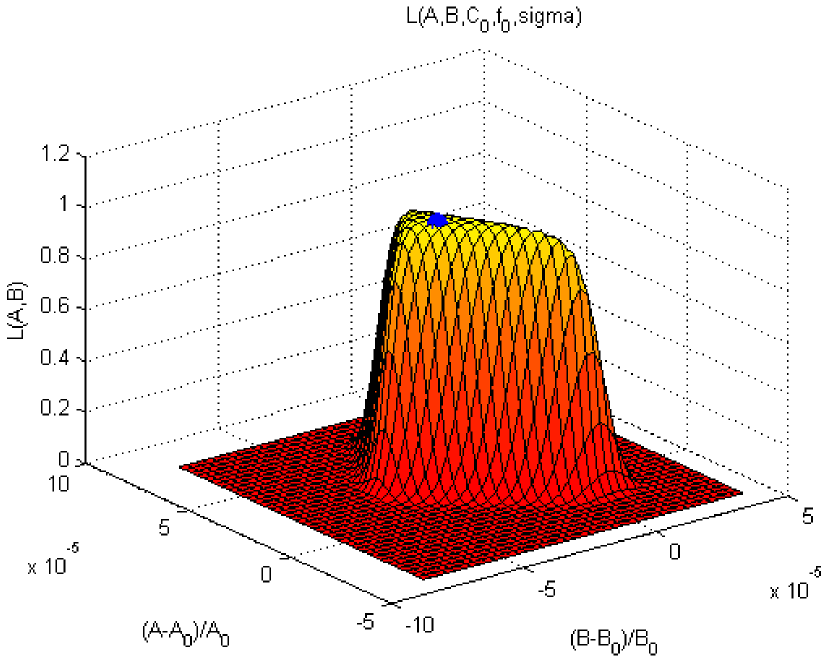

Figure 5.7: *Decreasing the noise deviation sharpens the likelihood function.*

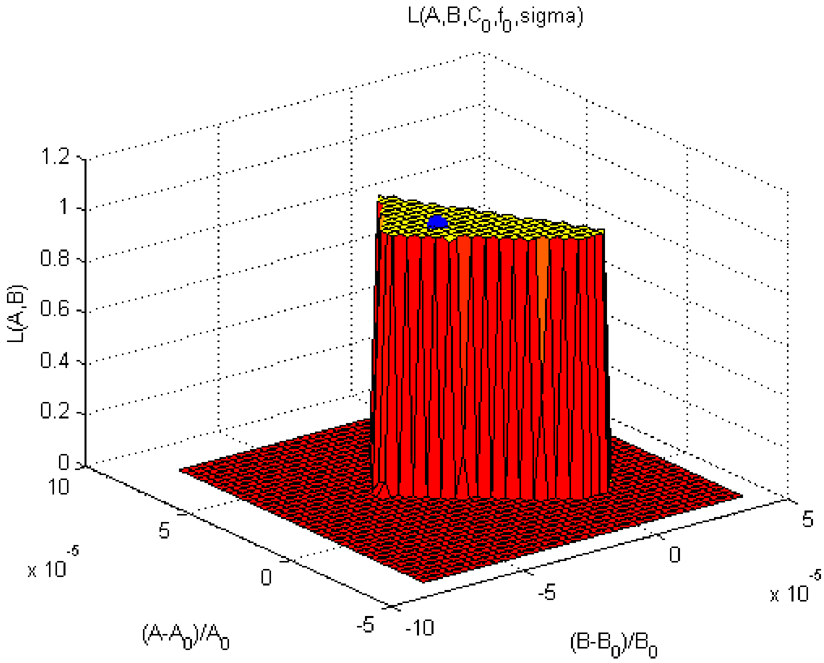

Figure 5.8: *Noise is decreased to arbitrary low value: the likelihood of the parameters reaches 1*

5.3 The effect of noise on the accuracy of estimators

The Cramér-Rao bound is the theoretical lower bound for the covariance of ML estimators. The covariance of estimators cannot be lower than the inverse of the Fisher-information. Fisher information is defined using second order partial derivatives of the log-likelihood function:

$$\mathbf{I}(\mathbf{p}) = -\mathrm{E}\left[\frac{\partial^2 \ln L(\mathbf{p})}{\partial^2 \mathbf{p}}\right] \tag{5.6}$$

As the cost function is the negative logarithm of the likelihood function, the Fisher information can be expressed using the Hess-matrix of the cost function.

$$\mathbf{I}(\mathbf{p}) = \mathrm{E}\begin{bmatrix} \frac{\partial^2 \mathrm{CF}}{\partial^2 A} & \frac{\partial^2 \mathrm{CF}}{\partial A \partial B} & \frac{\partial^2 \mathrm{CF}}{\partial A \partial C} & \frac{\partial^2 \mathrm{CF}}{\partial A \partial \theta} & \frac{\partial^2 \mathrm{CF}}{\partial A \partial \sigma} \\ \frac{\partial^2 \mathrm{CF}}{\partial B \partial A} & \frac{\partial^2 \mathrm{CF}}{\partial^2 B} & \frac{\partial^2 \mathrm{CF}}{\partial B \partial C} & \frac{\partial^2 \mathrm{CF}}{\partial B \partial \theta} & \frac{\partial^2 \mathrm{CF}}{\partial B \partial \sigma} \\ \frac{\partial^2 \mathrm{CF}}{\partial C \partial A} & \frac{\partial^2 \mathrm{CF}}{\partial C \partial B} & \frac{\partial^2 \mathrm{CF}}{\partial^2 C} & \frac{\partial^2 \mathrm{CF}}{\partial C \partial \theta} & \frac{\partial^2 \mathrm{CF}}{\partial C \partial \sigma} \\ \frac{\partial^2 \mathrm{CF}}{\partial \theta \partial A} & \frac{\partial^2 \mathrm{CF}}{\partial \theta \partial B} & \frac{\partial^2 \mathrm{CF}}{\partial \theta \partial C} & \frac{\partial^2 \mathrm{CF}}{\partial^2 \theta} & \frac{\partial^2 \mathrm{CF}}{\partial \theta \partial \sigma} \\ \frac{\partial^2 \mathrm{CF}}{\partial \sigma \partial A} & \frac{\partial^2 \mathrm{CF}}{\partial \sigma \partial B} & \frac{\partial^2 \mathrm{CF}}{\partial \sigma \partial C} & \frac{\partial^2 \mathrm{CF}}{\partial \sigma \partial \theta} & \frac{\partial^2 \mathrm{CF}}{\partial^2 \sigma} \end{bmatrix} \tag{5.7}$$

As Hess-matrix of the cost function is calculated in each iteration cycle to perform Levenberg-Marquardt step, Cramer-Rao bound can be estimated using the Fisher information matrices calculated in the last iteration cycle.

$$\mathbf{cov}(\mathbf{p}) \geq \mathbf{I}^{-1}(\mathbf{p}) \tag{5.8}$$

5.3.1 Case of low noise

As it is visible on figure 5.8, in case of low noise there is a set of parameter vectors with likelihood 1, and the likelihood of parameter vectors outside this set is 0. Each parameter vector with likelihood 1 is maximum likelihood estimator of the sine wave. Depending on the initial estimators and settings of the optimization algorithm, different parameter vectors with likelihood 1 can be reached. The Cramér-Rao bound cannot be calculated for this case: as the cost function is flat (constant 0) in the minima, the Hess-matrix is identically zero. Thus Fisher infromation matrix cannot be inverted and the theoretical lower bound for variance cannot be calculated.

5.3.2 Case of usual measurement noise

If measurement noise is high enough, shape of the likelihood function (and the cost function) is smoother: the curvature of the cost function near the minima can be calculated. This way it is possible to estimate the Fisher information (the expected value of Hess-matrices), and to calculate the Cramér-rao bound.

Chapter 6

Behaviour of ML estimators

Properties of maximum likelihood estimators are detailed in chapter 3. In the following sections experimental results are provided to observe behaviour of ML estimators used for ADC testing.

6.1 The method of observation

To examine the quality of estimators it is required to know the exact value of the given parameter. Investigating estimator properties such as consistency or variance is impossible without the theoretical value of the parameter estimated. In case of real measurements only other estimators are available regarding the signal parameters. Thus simulated measurements shall be performed to examine behaviour of estimators. The framework of investigation contains the following steps:

1. Chose a fix set of signal parameters. These will be used in the steps detailes above.
2. Perform multiple simulated measurements with the *same* signal parameters and the same *amount* of noise. As the realization of noise is different in each case, but signal parameters do not change, variance of estimators can be observed through these repeated measurements.
3. Create different sets of measurements, varying the number of sample, amount if noise, resolution, etc.
4. Process the simulated measurement record and store the estimators achieved.
5. Compare the calculated estimators to the real parameters: observe variance, consistency, etc.

6.2 Simulated measurements

Instead of using simulated measurement data created by myself, I have processed data provided by Ing. Jozef Liptak, PhD student of the Technical University of Kosice. In these simulations the device under test is an 8-bit nonlinear ADC. The static transfer characteristic has been derived from the measured transfer characteristic of the device NI-USB-6009 (see figure 6.1).

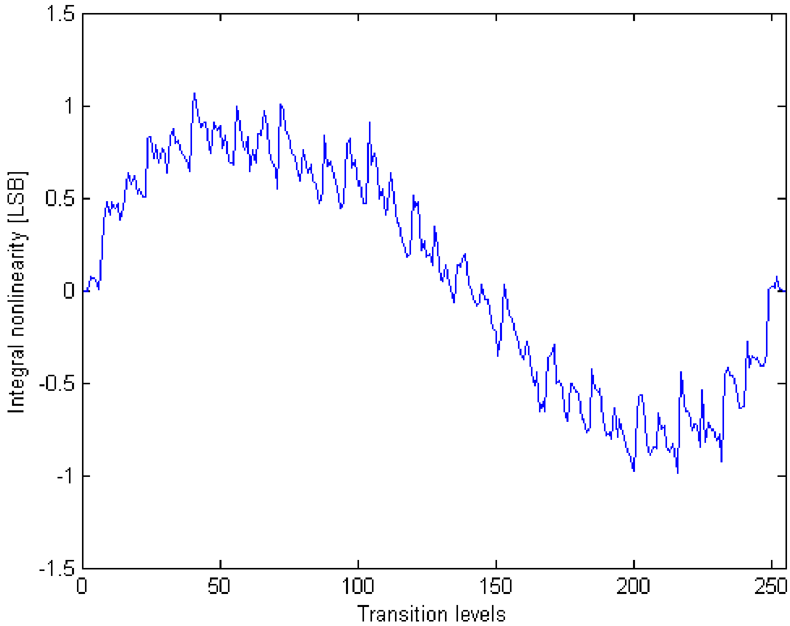

Figure 6.1: *Integral nonlinearity of the simulated ADC under test*

Amplitude, phase and DC component of the signal was fixed. Each set of simulated measurements contains 20 records, these records are results of simulations with identical signal parameters, and different realization of noise. To observe consitency and asymptotic behaviour of estimators, the number of samples increased from 1000 up to 100000. To ensure coherency, in each case 1 total period of the sine wave has been recorded, thus the frequency varied depending on the number of samples: $f/f_s = 1/M$, where M denotes the length of the record.

6.3 Estimation results

Seven sets of simulated measurement have been processed. The lengths of the records in these sets are 1000, 2000, 5000, 10000, 20000, 50000 and 100000 respectively. Twenty different data vectors have been processed in each set: thus expected values and variances of the estimators have been calculated. As both LS and ML estimators have been computed, behaviour of the two different types of estimators can be compared. To create readable illustrations, only three values have been drawn for each estimator: the expected value, and the $\mu + 3\sigma$ and $\mu - 3\sigma$ boundaries. Figures 6.2, 6.3, 6.4 and 6.5 displays the results achieved: the relative errors of estimators depending on the length of record.

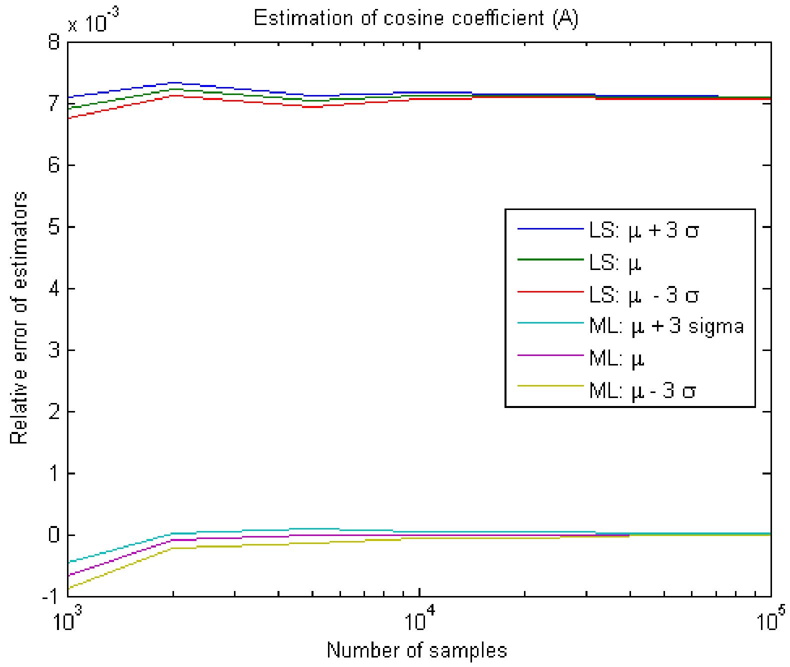

Figure 6.2: *ML and LS estimation of the cosine coefficient*

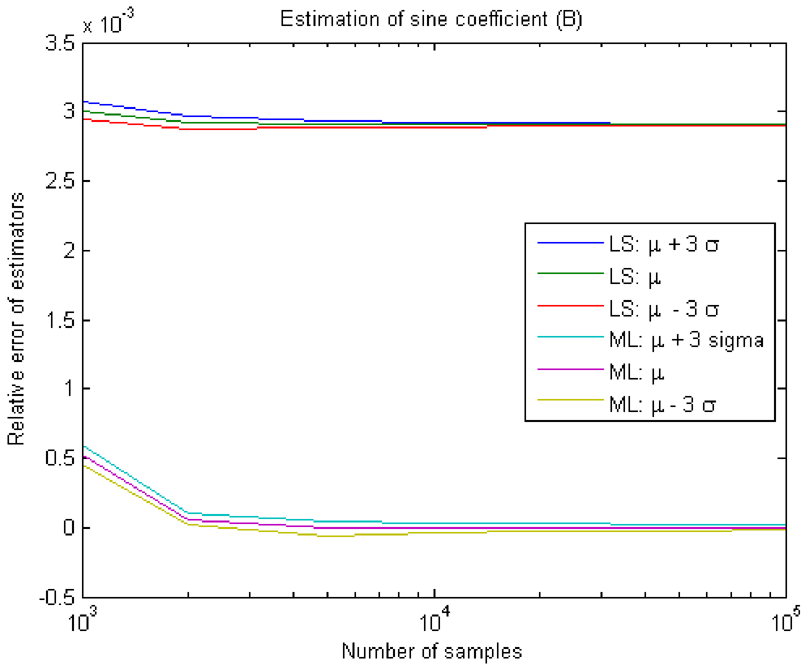

Figure 6.3: *ML and LS estimation of the sine coefficient*

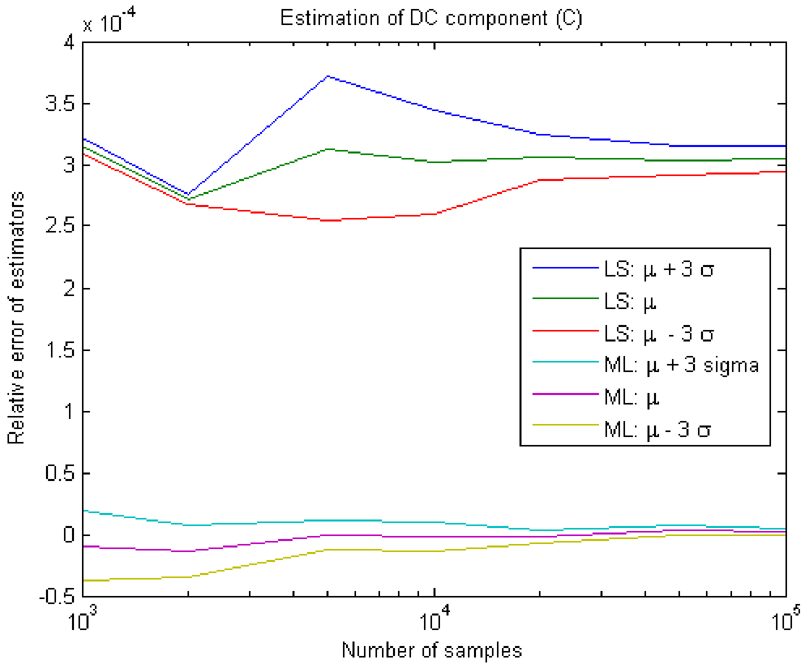

Figure 6.4: *ML and LS estimation of the DC component*

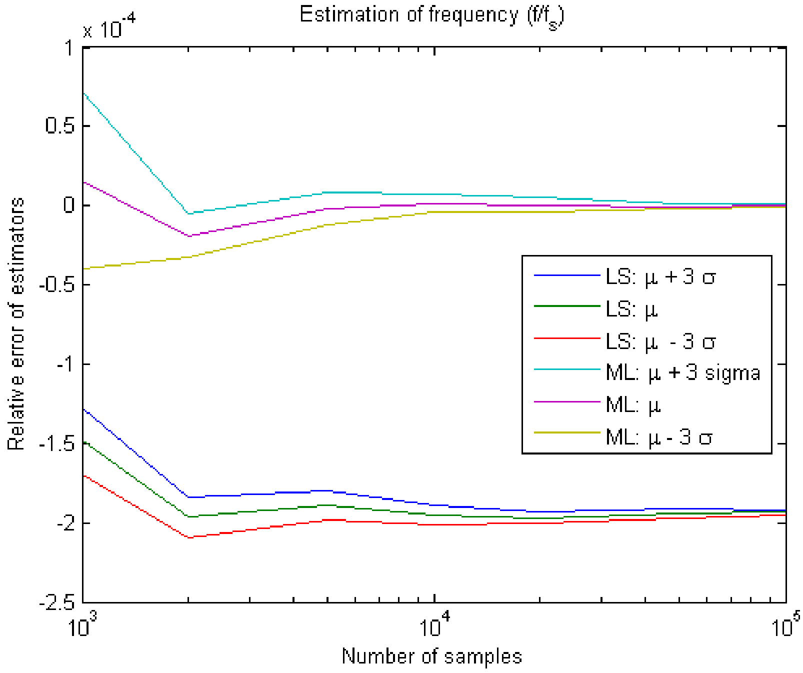

Figure 6.5: *ML and LS estimation of the sine wave frequency*

These results show the difference between LS and ML estimators: while LS estimator finds the best fitting sine wave in the output code domain, the nonlinearity of the converter misleads the LS estimators. ML estimators find the most likely sinewave that produces the recorded output: nonlinearity of ADC is handled in this model.

Chapter 7

Software tools for ADC testing

To perform ADC testing in practice, it is necessary to have software tools to automate the process of data acquisition, signal processing, and evaluation of measurement results. In this section two software tools are presented: one of them is a MATLAB toolbox for offline data processing, the other is a LabVIEW virtual instrument (VI). Both of them are developed by the author: though the ADC Test toolbox is based on the idea of the previous versions of ADC testing software developed by János Márkus, the functionality has been extended very largely, and the program code has been completely re-written (only the four parameter sine wave fit algorithm has been merged from the old toolbox). The LabVIEV VI contains very important MathScript codes written by Vilmos Pálfi, however data acquisition, dataflow programming, user interface and other MathScript codes have been developed by the author.

7.1 ADC Test toolbox for MATLAB

This program is a set of coherent MATLAB functions using Graphical User Interface (GUI). The goal of this software is to ease the process of ADC testing: using this envinronment, offline signal processing and measurement evaluation can be performed without any programming or deeper domain knowledge required. This toolbox provides the following main functionalities:

- Assembling measurement descriptors from acquired data and circumstances.
- Storing and handling measurement descriptors.
- Processing maeasurement records and evaluating results in multiple ways, such as
  - Sine wave fitting in least squares sense
  - Histogram testing using sine waves
  - Sine wave fitting using maximum likelihood estimation
  - Frequency domain analysis (FFT test)

7.1.1 Graphical user interface

The main window of the program is presented in figre 7.1. The functionalities appear separately on the screen. Information about the actual measurement descriptor is displayed in the center frame. On the top of the window, current measurement can be selected from the list of descriptors loaded, using a popup-menu. The data handling functionalities appear on the right side of the screen. Data processing possibilities are itemized in the lower frame of the main window.

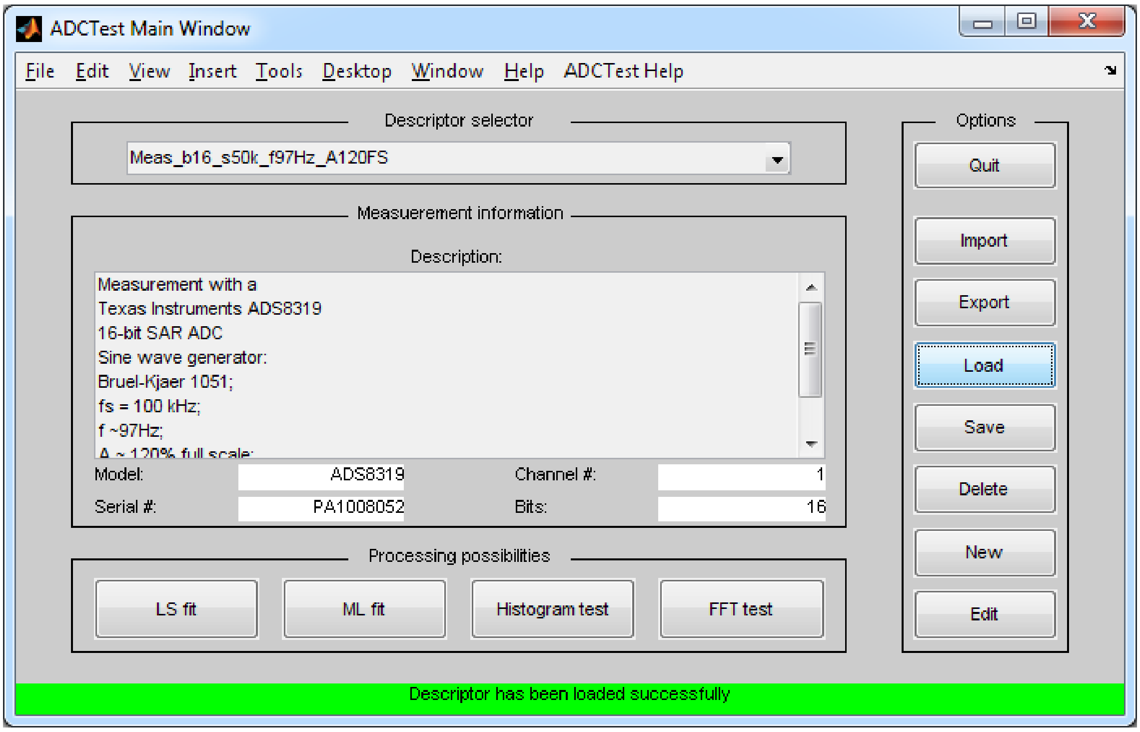

Figure 7.1: *Main window of the user interface*

7.1.2 Data handling functionalities

The toolbox provides the following options to handle measurement descriptors:

- **Import**: Importing entire measurement descriptors from the MATLAB workspace.
- **Export**: Exporting entire measurement descriptors to the MATLAB workspace.
- **Load**: Loading measurement descriptor from a specific XML file.
- **Save**: Saving measurement descriptor to a specific XML file.
- **Delete**: Removing actual measurement descriptor from the memory (does not affect files on disk).
- **New**: Creating new measurement descriptor: it can be assembled using the raw data (imported from the workspace) and the measurement circumstances.
- **Edit**: Editing existing measurement descriptor: more circumstances can be added, or incorrect information can be fixed.

The „New" option also allows the user to perform simulated measurements. This feature can be reached using the „Simulate measurement" pushbutton. Parameteres of the simulated measurement can be set such as the parameters of the excitation sine wave, the noise on the analog signal, and the transfer characteristic of the ADC under test. This characteristic can be previously measured, or assembled arificially. In the previous case the vector containing the INL values can be loaded from the MATLAB workspace, in the latter case the INL vector can be assembled from shapes (like sine wave or Hann window) and additive noise (like Gaussian or uniformly distributed).

7.1.3 Data processing possibilities

The main goal of the software tool is to process and evaluate measurements. There are multiple methods to process a measurement for ADC testing, in this toolbox four of them are implemented.

Four parameter sine wave fit in least squares sense

This dynamic test method has been developed to determine the effective number of bits (ENOB), and the signal to noise and distortion ratio (SINAD). The excitation signal is sinusoidal, thus a sine wave shall be fitted to the measurement record. The numerical methods used to perform the fit properly are described in section 2.1.3. The results are displayed in a dialog box, shown on figure 7.2. The estimated sine wave parameters and the calculated ADC parameters appear numerically, and the fitting residuals are displayed in two ways: statistical properties can be observed via the histogram, and Mod-T plot shows the location of the residuals in the phase space.

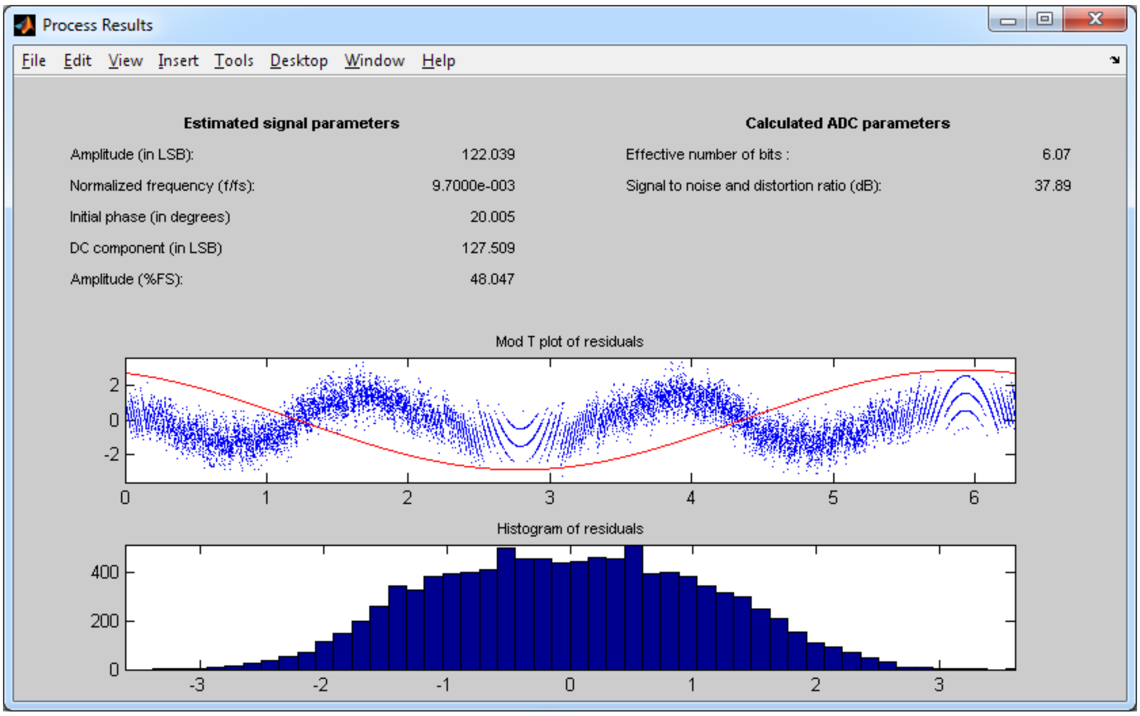

Figure 7.2: *Results window for LS fit*

Histogram test using sine waves

Estimation of static transfer characteristic of the ADC is also possible using sine waves. To perform histogram test properly, it is very important fo fulfil a few requirements regarding the settings of the excitation signal and the sampling. These requirements are itemized in section 4.2. This feature of the toolbox performs two tasks:

- Investigates whether the measurement record is appropriate for histogram testing or not.
- Calculates and displays estimators for integral and differential nonlinearity.

In case of inappropriate measurement records, warning meaasages appear to notice the user about the problems and the reasons (such as low number of samples, fractional periods, low signal amplitude, etc.). Figure 7.3 shows the estimated integral and differential nonlinearity of a 16-bit successive approximation ADC.

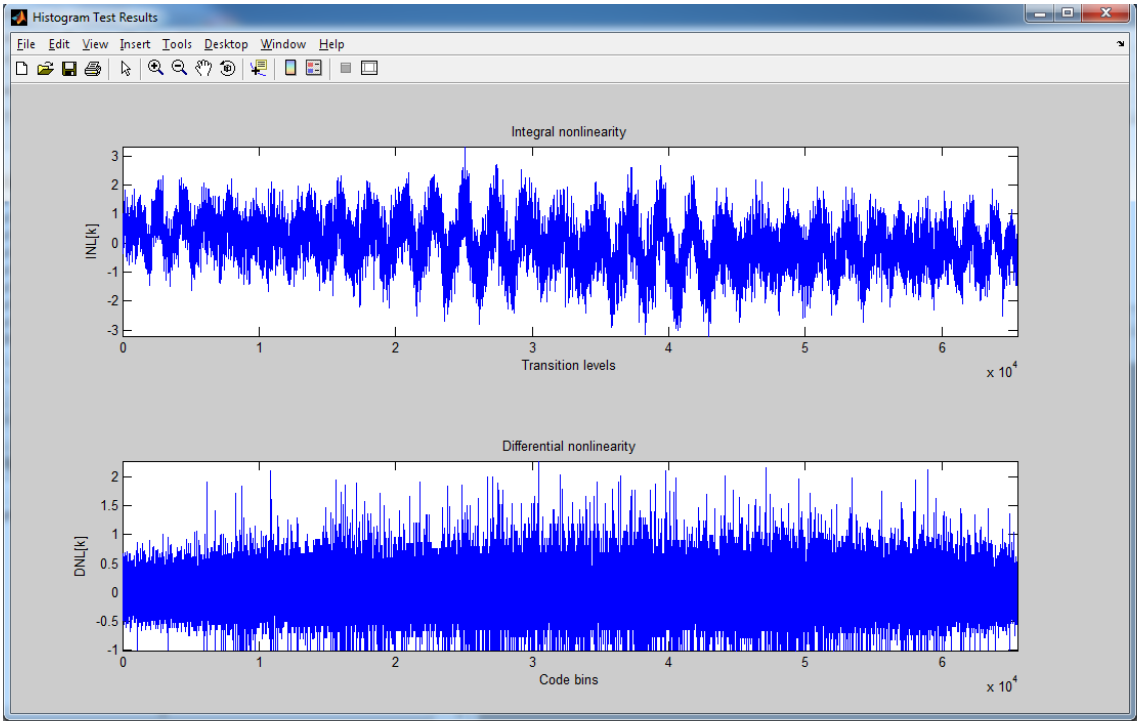

Figure 7.3: *Results window for histogram test*

FFT test

Frequency domain processing of a measurement record is also a useful option. Examining the DFT of the measurement record provides information about the harmonic distortion, the amount of noise and the spurious components. A very important quantity, the *Spurious-free dynamic range, SFDR* of the ADC under test can be calculated using the FFT of the record. In the results window (see figure 7.4), frequencies and magnitudes of the harmonic components appear. The SFDR is calculated relative to the carrier, and relative to the full scale (FS) of the converter. The amplitude spectrum of the record is also displayed from DC up to the Nyquist-freqency.

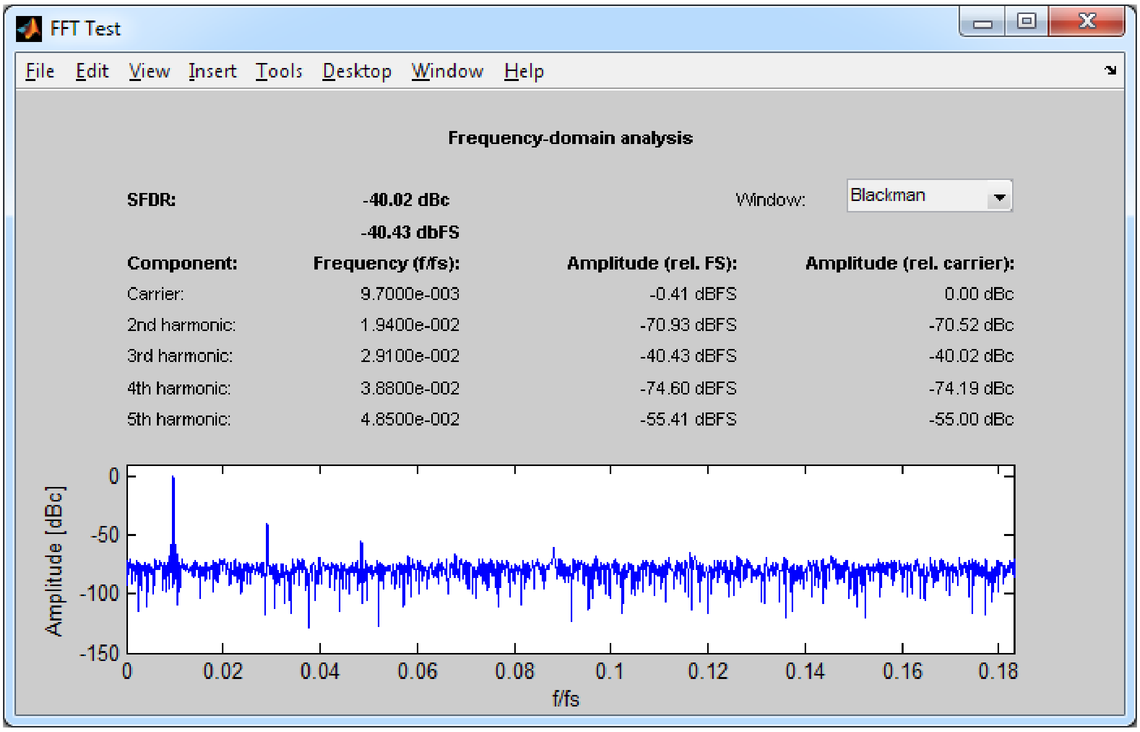

Figure 7.4: *Results window for FFT test*

In frequency domain analysis it is very important to avoid spectral leakage. In this toolbox three different windowing functions can be applied in time domain: Hann, Blackman, and 3-term Blackman-Harris windows are available. Naturally DFT can be performed without windowing, choosing the „None (rect)" option in the „Windowing" popup menu.

Sine wave fitting using maximum likelihood estimation

The most complex task implemented by the toolbox is the maximum likelihood estimation of sine wave parameters. Theoretical fundamentals, practical difficulties and solutions are detailed in the previous chapters. In this section ML estimation will be described only from the user's perspective. To perform estimation it is required to achieve initial estimators for both the excitation signal parameters and the code transition levels. These can be calculated using four parameter sine wave fit and histogram test. An initial estimator for the noise variance also shall be calculated as it is described in section 5.2. Then extrema of the likelihood function can be reached via numerical optimization. Figure 7.5 shows the framework of ML estimation.

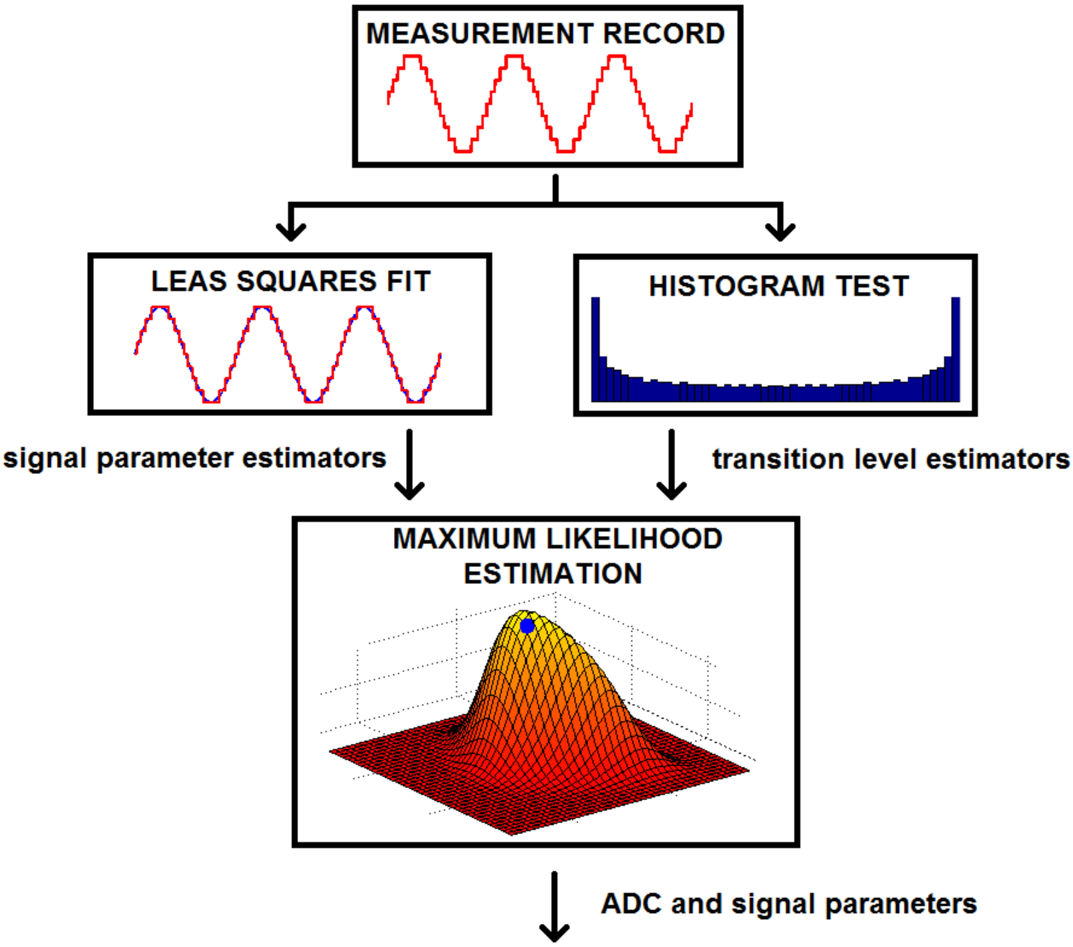

Figure 7.5: *Framework of ML parameter estimation*

Calling „ML fit" from the main window implicitly calls four parameter LS fit and histogram testing routines. Settings of LS fit shall be specified in a dialog box, and histogram test routine displays information (warnings, if necessary) regarding the appropriateness of the record. The numerical optimization can be followed via the GUI. As shown in figure 7.6, the actual values of the parameter estimators and the cost function are displayed in each iteration cycle. The scaling factor of the *Levenberg-Marquardt* step (λ) changes in each iteration, thus value of λ is also updated continuously. The user can specify the termination criteria for the optimization (such as termination tolerance, maximum number of iterations, or maximum number of cost function evaluations). Iteration can be stopped, paused and resumed using the pushbuttons on the right side of the window.

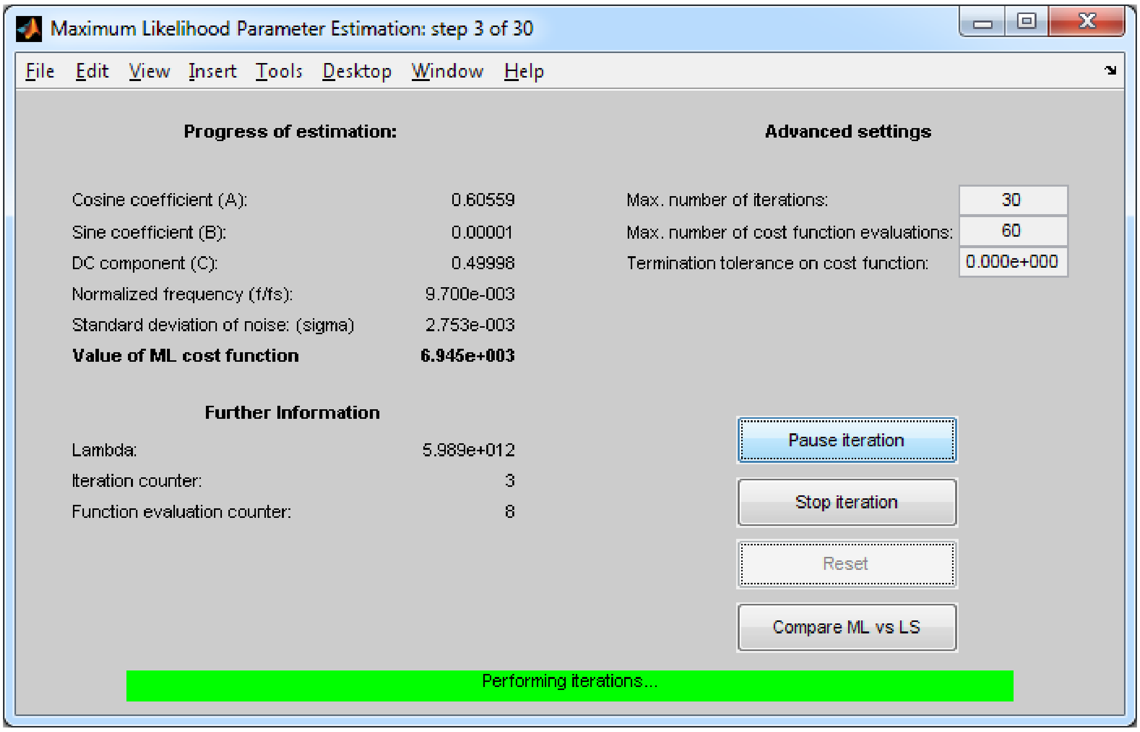

Figure 7.6: *Following iterations cycles of ML estimation*

Extremum of the cost function found by the optimization algorithm provides the ML estimators for the sine wave parameters. This way sine wave fit can be performed using the sine wave estimated in LS sense, and using the sine wave estimated via the maximum likelihood method. ADC parameters (datasheet quantities like ENOB and SINAD) can be calculated using both the ML and LS estimators. This way results of ML and LS fit can be compared easily in a comparison window (see figure 7.7). On the screenshot provided, ML and LS estimation is performed for a nonlinear ADC, thus differences in the fitted sine wave and the calculated device parameters can be observed. Note that the LS fit minimizes the mean squared value of residuals, thus minimizes the power of noise and distortion. This way SINAD and ENOB cannot be higher than their values calculated using LS estimators.

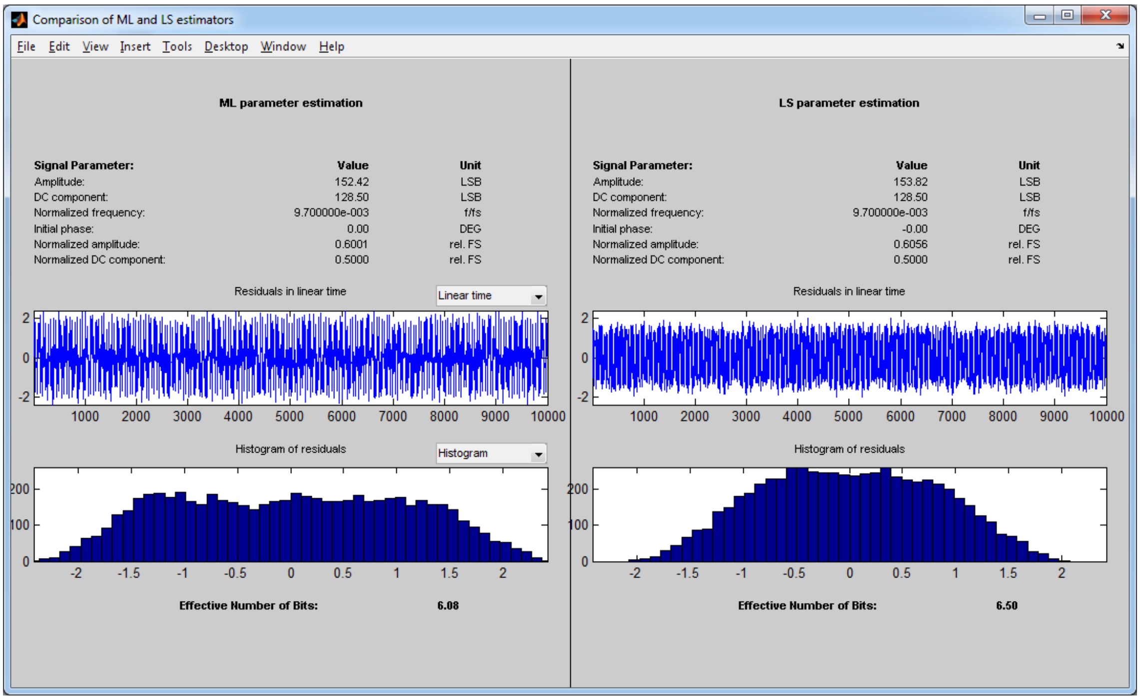

Figure 7.7: *Comparison of results using ML and LS estimators*

7.2 A LabVIEW tool to perform ADC testing

The LabVIEW environment provides advanced software interfaces to measurement and data acquisition devices. The MathScript module also allows the developer to implement complex signal processing algorithms in a text-based programming language. This way excitation signal generation, data acquisition, and data processing can be integrated into one software component: a virtual instrument (VI). The front panel of the VI (see figure 7.8) provides the user interface. On the front panel

- settings of the signal generator (DAC) and parameters of the sine wave generated can be specified,
- settings of data acquisition (sample rate, length of record) can be specified,
- data acquisition can be performed and repeated using a pushbutton,
- data processing results are displayed numerically and graphically.

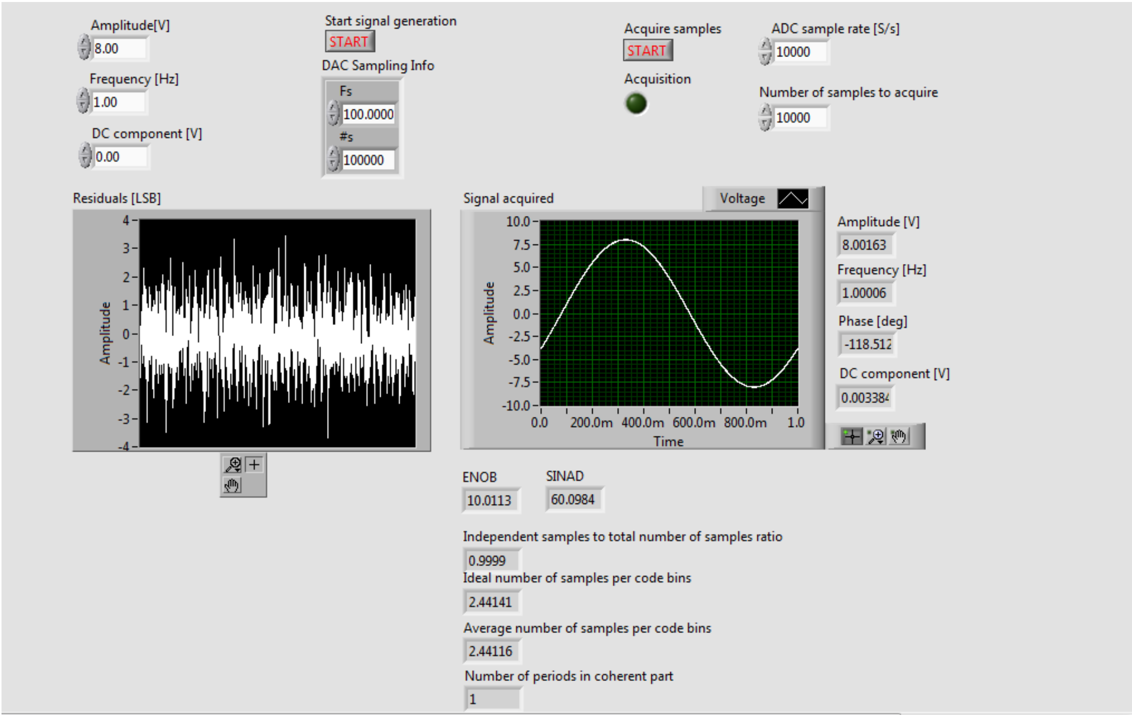

Figure 7.8: *Front panel of the virtual instrument*

To generate analog excitation signal, this VI uses an NI 9263 analog output module containing a 16-bit DAC. The data acquisition device is an NI 9201 analog input module containg a 12-bit ADC. This ADC is the device under test in this project. To write scalable and maintainable code, the complex signal processing algorithms are encapsulated in subVIs. Each subVI performs a general task like sine wave fitting, coherency investigation or INL estimation. These elements can be called from multiple different top level VIs depending on the purpose of the application.

Chapter 8

Conclusion

This report is focusing on the practical issues of ADC testing using ML parameter estimation. To introduce this topic it is necessary to motivate the usage of the ML method. Thus currently standardized and used techniques are also described in this paper. The first three chapters present the state of art, the other chapters contain the results of my research efforts. These results can be summarized in the following points:

- Maximum likelihood estimation can be performed despite of the problems detailed in chapter 4 and chapter 5, using the solutions proposed in those sections.
- Properties of ML estimators have been examined via simulated measurements.
- The ML method has been implemented in MATLAB and LabVIEW envinronments. The MATLAB toolbox is available on the web [14]. The LabVIEW VI performs excitation signal generation, data acquisition and signal processing in one software unit.

The majority of this study deals with the challenges and difficulties of the realization. The most important outcome of this research is that maximum likelihood estimation of ADC parameters is feasible, and does not require excessive efforts. However, the process of measurement evaluation is more complex than that of the standard methods. To calculate ML estimators properly, it is necessary to perform supporting computations like examination of coherence, histogram testing and four parameter sine wave fit in LS sense. Nevertheless the entire framework of ML estimation can be integrated into one software tool (see chapter 7), and measurement evaluation can be performed with acceptable time consumption. For a usual measurement record (from 10000 up to 1 million samples, resolution from 12 up to 16 bits), total computation (from the raw measurement data to the calculated datasheet quantities) takes a few or a few tens of seconds.

Sine wave fit using maximum likelihood estimation is realizable, and provides more accurate datasheet quantities than the ones calculated using standard methods. Using ML parameter estimation for ADC testing is recommended in each case, when it is worth making more efforts to increase the precision.

Chapter 9

Outlook

Using maximum likelihood estimation for ADC testing is a new and yet unusual method to examine quality of analog-to-digital converters. Researchers working in this area have developed the theoretical background of this method, and investigated the properties of ML estimation. The results are partly achieved via simulations, because examining a parameter estimation method without the exact value of the model parameters is impossible. In case of simulated measurements some of the nonideal phenomena are modeled, but all kinds of disturbances cannot be described and simulated properly in one model. Processing real measurement records highliths excellently the practical challenges that appear in ADC testing using ML estimation. Most of the problems investigated in the previous chapters occured when real measurement records were processed. The following aims are attractive for further efforts:

- Describing nonideal phenomena appearing in measurements.
- Extending the model to handle nonideal sampling (jitter).
- Decreasing the computation time of numerical optimization:
  - Implementing the core of the algorithm in lower level computing language (e.g. C/C++ instead of MATLAB or LabVIEW).
  - Parallelizing the algorithm wherever it is possible. This way computation time can be improved significantly using technologies like OpenMP API or general purpose GPUs.
- Improving the robustness of the algorithm to be prepared to handle almost all the problems that can occur in real measurement.

The topics itemized above are closely related to the questions that are raised and largely answered in this report. Thus examination of the problems collected in this outlook can be based on the results described in the previous chapters.

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