Examples in the theory of continuous trading with friction

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Abstract. In this study we have constructed examples which show that in illiquid markets (where trading costs are a superlinear function of trading speed) it is more difficult to create arbitrage then in models where transaction costs are linear (or there are no such costs at all). We have achived this by constructing price processes which allow riskless profit if trades can be executed infinitely fast but fininte speed misses these opportunities.

Our results help to clarify differences between various trading mechanisms in terms of the theory of arbitrage.

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1. Introduction

In financial markets, trading activity moves prices against the trader, that is, buying increases execution prices and selling decreases them. This property of the market is known as illiquidity, market depth or price-impact, opposed to a state of the market where an asset can be sold or bought easily at arbitrary quantities.

Illiquidity is a phenomenon of the market that an asset cannot be easily sold or bought without a substantial loss in value. Illiquid assets may also be hard to sell quickly because of a lack of ready and willing investors or speculators to purchase the asset. These direct and indirect costs associated with the execution of financial transactions are called friction, an empirically well documented phenomena.

In frictionless models prices are independent of the speed of trading, and are the same for any amount traded. Loosely speaking, in a frictionless setup, one can perform transactions at an infinite rate. The models where friction is present depart heavily from the literature on frictionless markets. One type of friction is due to proportional transaction costs, where prices differ for buying and selling, but are not sensitive to quantities. With this kind of friction in the model one has to expect a loss that is proportional to the quantity bought or sold, hence we call this linear friction. Illiquidity, in general, generates a more agressive type of friction, called superlinear friction, where quantites traded give rise to a loss that prevents buying or selling at infinite rates.

In a frictionless model the number of shares are represented by an arbitrary predictable process, while in a model with transactional costs the number of shares is assumed to have finite variation. With illiquidity an even narrower class has to be used, that is, strategies are absolutely continuous by hypothesis. This is explained by the following arguments.

In the presence of illiquidity execution prices become unfavorable as traded quantities grow, and thus buying or selling too fast becomes impossible. As a result, trading is feasible at finite rates only. This feature sets apart models with illiquidity from frictionless markets, and also from proportional transaction costs models.

Fundamental question about financial models have different answeres according to different setups, resulting, for example, in different arbitrage criteria. In the frictionless case arbitrage is characterised e.g. in S. E. Shreve [5], that is if the existance of an equivalent martingale measure is guaranted, then there is no arbitrage opportunity. In models where transaction costs are present, there is arbitrage if and only if the underlying price process can be approximated with a martingale uniformly, characterised by Guasoni-Rásonyi-Schachermayer [6]. In a continuous time setup with illiquidity, superhedging and absence of arbitrage are characterised by M. Rásonyi and P. Guasoni [2]. In this publication powerful tools have been developed that are capable of detecting an arbitrage opportunity. Loosely speaking, there is no arbitrage opportunity, provided that a measure and a martingale can be constructed so that the average distance of the price process and the constructed martingale as well as the total variation distance of the original and the new measure can be made arbitrarily small.

In the context of arbitrage, the difference between these models in terms of price processes has not yet been clarified. It has been shown in Guasoni-Rásonyi-Schachermayer [3], that there are price processes that generate arbitrage opportunity in frictionless setup, but do not do so with transaction costs. However, examples of price processes that yield no arbitrage in the presence of illiquidity, but generate arbitrage opportunity both in the frictionless case and with transaction costs have not yet been presented. This work aimes at filling the gap. Such examples are important beceause they highlight the differences between various setups in a practical and spectacular way, making the deeper level price system mechanics visible.

The rest of the paper proceeds with Section 2, presenting a mathematical introduction and setting the scene for the construction of examples by introducing the model and enumerating the tools that are needed in the procedure. A first example is developed in Section 3 for absence of arbitrage of the second kind. In Section 4 the more natural concept of absence of arbitrage is treated and a class of more involved examples is presented.

2. Model

To set the scene, from now on we present the concepts emerging in the publication M. Rasonyi and P. Guasoni [2], R. Almgren and N. Chriss [1] and L. Rogers and S. Singh [4].

For a given filtration $(\mathcal{F}_t)_{t\geq 0}$ we say that the function $f : \Omega \times [0, T] \mapsto \mathbb{R}$ is progressively measurable if $f|_{[0,t]\times\Omega}$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ measurable, for all $t \in (0, T]$, where $\mathcal{B}([0, t])$ denotes the Borel sigma-algebra generated by the open sets of $[0, t]$. Let (M, d) be a metric space then with $E \subset \mathbb{R}$ we say that the function $f : E \mapsto M$ is càdlàg (right continuous with left limits) if for all $t \in E$ the left limit $\lim_{s\uparrow t} f(t)$ exists and the right limit $\lim_{s\downarrow t} f(t)$ also exists and this latter equals the value $f(t)$.

We start to introduce the model by fixing a finite time horizon $T > 0$ and an appropriate filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ being right continuous, i.e. $\cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$, and let $\mathcal{F}_0$ be the trivial σ-algebra. All the random processes below are defined on this probability space. Dependence on $\omega \in \Omega$ will be omitted in notation. The progressively measurable random function $\phi : \Omega \times [0, T] \mapsto \mathbb{R}$ will denote a stochastic process representing a trading strategy, i.e. the frequency of our selling or buying. Thus by integrating this process over a certain time period yields the accumulated number of stocks we hold. The cumulative number of shares bought or sold on a finite time period is given for all $t \in [0, T]$ with $\int_0^t |\phi_u|du$ and the number of shares we are holding at time $t \in [0, T]$ is $\int_0^t \phi_u du$. If for $t \in [0, T]$ fixed $\phi_t < 0$, it means that we are selling the stock or, technically speaking, we are going short with rate ϕ_t. On the other hand if $\phi_t > 0$ we likewise say that we are buying or, technically speaking, going long with rate ϕ_t.

We will consider a stochastic process $(S_t)_{t\in[0,T]}$, which will represent the price of the stock we are trading. This underlying stock price process will be by hypothesis càdlàg. The fixed real numbers z_0 and v_0 will play the role of the initial capital on our bank account and the initial quantity of shares we hold respectively.

In the presence of illiquidity for a given strategy ϕ, friction reduces the cash positions by making purchases more expensive, and sales less profitable. This effect will be modeled by the non-negative and convex (in its third variable) function $G : \Omega \times [0,T] \times \mathbb{R} \mapsto \mathbb{R}^+$, with the further assumptions that $G_t(x) \geq G_t(0)$ and there exists $\alpha > 0$ and a cadlag process $(H_t)_{t\in[0,T]}$ such that

Assumption 2.1.

$$\begin{aligned}
&\inf_{t\in[0,T]} H_t > 0 \quad a.s.\\
&G_t(x) \geq H_t|x|^\alpha \quad \textit{for all } x \in \mathbb{R}, \qquad (1)\\
&\int_0^T \sup_{|x|\leq N} G_t(x)dt < \infty \quad a.s. \quad \textit{for all } N > 0.
\end{aligned}$$

Assumption 2.1 will be referred to as the superlinearity assumption. Although the function G could take many forms satisfying the previous assumption, in the corresponding literature there are two frequently used settings. The function G could depend on ω through the price S_t taking the form $G_t(x) = \lambda S_t|x|^\alpha$ or, as it is in our setup, it will be of the form $G_t(x) = G_t(x; \lambda, \alpha) = \lambda|x|^\alpha$, with $\alpha > 0$ and $H_t := \lambda > 0$ for all $t \in [0,T]$, it will also be non-random and will depend on time through its argument ϕ_t. With these definitions, with a given strategy ϕ, at time $t \in [0,T]$ position in the risky asset V_t and in the cash account X_t are defined as

$$V_t(v_0,\phi) := v_0 + \int_0^t \phi_u du, \qquad (2)$$

$$X_t(z_0,\phi) := z_0 - \int_0^t \phi_u S_u du - \int_0^t G_u(\phi_u)du. \qquad (3)$$

The complete dynamics of the portfolio model is given by the equations above. The value $V_t(v_0,\phi)$ is the amount of shares in our hand at time t, the sum of the initial number of shares and subsequent flows. If we pool the integrals in (3) and observe the expression $\int_0^t \phi_u(S_u + \frac{G_u(\phi_u)}{\phi_u})du$ with ϕ assumed non-zero, we see that the actual price we pay at time t for the stock is not S_t but instead, as a result of friction, the value $S_t + \frac{G_t(\phi_t)}{\phi_t}$. The value S_t can be called the "hypothetical price" of the underlying stock and illiquidity results in this price being biased by the term $\frac{G_t(\phi_t)}{\phi_t}$. Note also, that due to this bias, the bank account equity (3) can be arbitrarily negative, even when $\phi_t < 0$. The quantity $S_t + \frac{G_t(\phi_t)}{\phi_t}$ can be called the instantaneous execution price which is - relative to S_t - higher when buying and lower when selling. The last term in equation (3) summarises the impact of trading on execution prices. The condition $G_t(x) \geq G_t(0)$ means that inactivity in trading is always cheaper than any buying or selling. In the present paper $G_t(0) = 0$ will hold.

Price dynamics can be classified through the value of the variables α, λ and hypothesis on the function ϕ. Different parameters and assumptions on the strategy yield different models, with the same question to be answered, namely whether there is an arbitrage opportunity. Arbitrage exists as a result of market inefficiencies. It is the possibility of a risk-free profit through taking advantage of differences in price of a single asset. Depending on α and the specification of ϕ with the same fixed price process there are different criteria for the absence of arbitrage. This way it is possible that the same price process yields different aswers to our questions according to the setup. Assumptions on the function ϕ change along with the variable α. Typically the situation gets better in terms of arbitrage as α increases and the assumptions on the strategy ϕ are getting stronger, meaning that there are weaker and weaker conditions guaranteeing the absence of arbitrage.

We will construct price processes that yield arbitrage in one particular model of friction but not in another. To continue, some ideas are necessary to be introduced. For better understanding of the significance of our work we will try to emphasize the contrast between arbitrage criteria in different models, i.e. the frictionless case, the transaction cost model and the superlinear illiquidity model, all introduced below. The theorems corresponding to superlinear setup will be treated with rigor, but the criteria and sufficient conditions relevant to other setups are only mentioned to illuminate the gap between these models.

Definition 2.2 (Superhedgebility)**.** *For fixed $z_0 \in \mathbb{R}$ initial capital we say that the scalar random variable W is superhedgeble with respect to a strategy ϕ if $X_T \geq W$ and $V_T \geq 0$ hold almost surely.*

Definition 2.3 (Arbitrage of the first kind)**.** *If for $v_0 = 0$ and $z_0 = 0$ there exists a trading strategy ϕ and an almost surely non-negative random variable W with $P(W > 0) > 0$ that is superhedgeble, then we say that there is an arbitrage opportunity of the first kind. In the absence of such strategy we say that there is no arbitrage of the first kind. We will refer to this condition as (NA).*

Definition 2.4 (Arbitrage of the second kind)**.** *If for $v_0 = 0$ there is an initial capital $z_0 < 0$ and a trading strategy ϕ such that $W = 0$ is superhedgeble, then we say that there is an arbitrage opportunity of the second* *kind. In the absence of such strategy we say that there is no arbitrage of the second kind. We will refer to this condition as (NA2).*

Remark 2.5. *In Definition 2.4 setting* $z_0 := 0$ *and* $W := w$ *with some constant* $w > 0$ *would lead to the same notion.*

The idea behind these definitions is that arbitrage is the existence of an opportunity that starting with no shares in our hands and a debt on the bank account, we can invent and employ such a clever strategy that after some time we will surely have a positive bank account and we will hold no shares, i.e. fortune comes into existence out of nothing. It is clear that (NA) implies $(NA2)$, but the opposite is not true.

Remark 2.6. *It is equivalent to say that there is no arbitrage opportunity of the second kind if for* $v_0 = 0$ *and for all* $z_0 < 0$ *the degenerate random variable* $W = 0$ *is not superhedgeble.*

The fact that $\lambda = 0$ boils down to the frictionless case can easily be seen if ϕ is integrable and has a primitive function Φ, because setting $z_0 = 0$ and employing a formal integration by parts on equation (3) yields $X_T(0, \phi) = \int_0^T \Phi_u dS_u$. This formula requires essentially no conditions on Φ - which means that neither absolute continuity nor the bounded variation property is prescribed for the strategy, and only square integrability is required. This leads to the classical frictionless market. Due to the absence of restrictions on ϕ one can perform trading with infinite speed if that is advantageous for the bank account. We can say that there is no illiquidity in this model.

In the frictionless case the price is often modelled by an Ito process. In a market model without friction, classical stochastic calculus results along with Girsanov's theorem and the well known equivalent martingale measure arbitrage criteria -stated in Remark 2.7 below- hold.

Remark 2.7. *In the frictionless case, if there exists an equivalent measure such that* S_t *is a martingale with respect to it, then (NA) holds. In the case of discrete time, even equivalence can be stated.*

When $\alpha = 1$, a market model emerges where transaction costs are present through the linear function $G(x) = \lambda|x|$, which represents the cost of participating in a market (i.e. money paid to brokers for example). For details of price dynamics and more see Guasoni and Rásonyi and Schachermayer [3]. In this model "linear illiquidity" is present. Related literature allows not only absolute continuous strategies Φ but arbitrary Φ with almost surely bounded variation.

Now, say, we have a price process $(S_t)_{t\in[0,1]}$ such that $S_0 = 2$ and $S_T = 1$. The strategy Φ in this model can have jumps, so observe the strategy

$$\Phi_t = -\mathbb{1}_{[0,1)}(t) \tag{4}$$

in other words $\Phi_{0-} := 0$, $\Phi_0 := -1$, $\Phi_t := -1$ for $t \in (0, 1)$ and $\Phi_1 := 0$. In plain English, we execute transactions with infinite speed, going short with one unit of shares at time $t = 0$ and selling it immediately at the end. At time zero we gain 2 units of money for going short, and at the end we pay only 1 unit of money, making 1 unit of profit out of the difference with zero risk, that is there exists an opportunity of arbitrage which we have just exploited with the latter strategy. In our model -introduced below- this same strategy creates no arbitrage due to the presence of friction, penalizing infinite speed transactions with unbounded losses. That is, we do not allow such a strategy to be utilized.

Remark 2.8. *The above example of strategy also works if the price process, instead of going from 2 to 1, simply sinks from 1 to zero. The main idea is that such a deterministic change, regardless of the direction of the change, can be exploited to create riskless profit.*

Remark 2.9. *In the transaction cost model with linear friction for fixed λ, where λ is the first parameter of the function G, if there is an equivalent measure Q and a martingale M_t with respect to it such that $\sup_t |M_t - S_t| < \lambda$, then (NA) holds. There is even a kind of equivalence, see [6] for details.*

In our model, which is characterised by $\alpha > 1$ and ϕ absolutely continuous, i.e. a member of the class $\mathcal{C}$ defined below, yields "superlinear illiquidity" with the function representing friction taking the form $G(x) = \lambda|x|^\alpha$. For simplicity we will work with $\alpha = 2$ and $\lambda = 1/4$. In this model the cumulative number of shares are , by assumption, finite almost surely. We will say that a strategy ϕ is feasible if it is a member of the class

$$\mathcal{C} := \{\phi : \phi \text{ is } \mathbb{R}\text{-valued, progressively measurable, } \int_0^T |\phi_u| du < \infty\}.$$

From now on every strategy ϕ will be assumed feasible, i.e. $\phi \in \mathcal{C}$ holds, unless otherwise stated. With superlinear friction it is clear that one can not execute selling or buying at infinite rate, because the losses generated by high speed trading can be infinite, even when the price favors such a move from the trader. It can happen that, figuratively speaking, there is not enough time to exploit certain properties of a particular price process. A fact that will be employed in this dissertation to construct examples.

By Remark 2.6 one can construct a useful criterion for the condition (NA2) to hold, which we will formulate in the next proposition.

Proposition 2.10. *The condition (NA2) holds if and only if the superhedgebility of $W = 0$ for a fixed initial capital z_0 with respect to some strategy ϕ implies $z_0 \geq 0$.*

Now we will introduce ideas from M. Rasonyi and P. Guasoni [2] to be able to state a sufficient condition for (NA2), Theorem 2.12 below, which is only applicable with superlinear friction. The notion of Legendre conjugate will be needed in the followings so we introduce it now. The Legendre conjugate of the function $f : \mathbb{R} \mapsto \mathbb{R}$ is denoted by f^* and defined for $y \in \mathbb{R}$ as

$$f^*(y) := \sup_{x\in\mathbb{R}}(xy - f(x)).$$

Let us introduce a class of probability measures. Fix a real number $1 < \beta < \alpha$, where α is as in (2.1). Let $\mathcal{P}$ denote the set of probabilities Q equivalent to P such that

$$E_Q \int_0^T H_t^{\frac{\beta}{\beta-\alpha}} (1 + |S_t|)^{\frac{\beta\alpha}{\alpha-\beta}} dt < \infty, \tag{5}$$

where H_t is as in (2.1). We set $\beta := \frac{1+\alpha}{2}$. In our simple setup, with $\alpha = 2$ and $H_t = \lambda = 1$ (5) takes the form

$$E_Q \int_0^T (1 + |S_t|)^6 dt < \infty.$$

Now we quote a special case of a result in M. Rasonyi and P. Guasoni [2, Corollary 3.13] in one dimensional settings. We will use this with some modification considering the remark M. Rasonyi and P. Guasoni [2, Remark 3.12].

Theorem 2.11 (Superhedging theorem)**.** *Let W be a scalar random variable, $z_0 \in \mathbb{R}$ and suppose that superlinearity assumtion (2.1) holds. Then there exists a feasible strategy $\phi \in \mathcal{C}$ such that $X_T(z_0, \phi) \geq W$ and $V_T(0, \phi) \geq 0$ almost surely if and only if*

$$z_0 \geq E_Q(W) - E_Q \int_0^T G_t^*(Z_t - S_t)dt, \tag{6}$$

for all $Q \in \mathcal{P}$ and for all non-negative Q-martingales $(Z_t)_{t\geq 0}$, such that $E_Q|Z_T|^\gamma < \infty$, where γ is such that $\frac{1}{\beta} + \frac{1}{\gamma} = 1$, where β is as in (5) and $E_Q|Z_T W| < \infty$.

As mentioned above, for simplicity we will set $G_t(x) := \frac{x^2}{4}$, i.e. $\lambda = 1/4, \alpha = 2$, and we should note here that this quadratic case will be very convenient from several aspects of which one is that this quadratic function under the previously introduced transformation becomes $G^*(x) = x^2$. This way (6) takes the form

$$z_0 \geq \sup_{Q,Z} \Big[E_Q(W) - E_Q \int_0^T |Z_t - S_t|^2 dt \Big], \tag{7}$$

where $Q \in \mathcal{P}$ and Z ranges over all Q-martingales with the same properties as in Theorem 2.11.

We should also note that choosing $\alpha > 1$ differently would not matter in any notable way, that is we could say that there is no loss of generality in our simple setup.

The following sufficient condition emerging from the Superhedging theorem needs a remarkably weak hypothesis and will be effective in verifying the condition (NA2) for a broad class of price processes.

Theorem 2.12 (A sufficient condition for (NA2)). *If for all $\varepsilon > 0$ there exists a probability measure $Q = Q(\varepsilon) \in \mathcal{P}$ and a Q-martingale $(Z_t = Z_t(\varepsilon))_{t\geq 0}$ such that $E_Q|Z_T|^{\frac{\beta}{\beta-1}} < \infty$, where β is as in (5), and*

$$E_Q \int_0^T |Z_t - S_t|^2 dt < \varepsilon. \tag{8}$$

then (NA2) holds.

Proof. Attempting the use of Proposition 2.10 let us assume that for the fixed initial capital z_0 the random variable $W = 0$ is superhedgable with some strategy. Using the Superhedging theorem Theorem 2.11 we have that for all measures $Q \in \mathcal{P}$ and Q-martingales $(Z_t)_{t\geq 0}$ such that $E_Q|Z_T|^\gamma < \infty$, and γ is such that $\frac{1}{\beta} + \frac{1}{\gamma} = 1$, where β is as in (5) the inequality

$$z_0 \geq -E_Q \int_0^T |Z_t - S_t|^2 dt$$

holds. Now by assuming (8) for fixed $\varepsilon > 0$ we have that

$$z_0 \geq -E_{Q^{(\varepsilon)}} \int_0^T |Z_t^{(\varepsilon)} - S_t|^2 dt > -\varepsilon$$

Letting ε to zero yields $z_0 > 0$ and this by Proposition 2.10 completes the proof. □

We proceed by introducing a sufficient condition for (NA). The only difference between this and the sufficient condition for (NA2) is that here we have an additional requirement, that is the equivalent measure must be close to the original measure in the sense of total variation. The total variation distance of two measures is defined by $||P - Q||_{tv} := \sup_{A\in\mathcal{A}} |P(A) - Q(A)|$. It is well known that the total variation is calculated as $\frac{1}{2}\sum_{\omega\in\Omega} |P(\omega) - Q(\omega)|$ if Ω is discrete and $E|\frac{dQ}{dP} - 1|$ otherwise.

Theorem 2.13 (A sufficient condition for (NA)). *If for all $\varepsilon > 0$ there exists a probability measure $Q = Q(\varepsilon) \in \mathcal{P}$ and a Q-martingale $(Z_t = Z_t(\varepsilon))_{t\geq 0}$ such that $E_Q|Z_T|^{\frac{\beta}{\beta-1}} < \infty$, where β is as in (5),*

$$E_Q \int_0^T |Z_t - S_t|^2 dt < \varepsilon \quad \text{and} \quad E|\frac{dQ}{dP} - 1| < \varepsilon, \tag{9}$$

then (NA) holds.

Proof. Assume that the hypothesis of the theorem holds and suppose there is arbitrage of the first kind, i.e. there exists an almost surely positive random variable Y with $P(Y > 0) > 0$ that is superhedgable, i.e. for which $X_T(0, \phi) \geq Y$ and $V_T(0, \phi) = 0$ hold almost surely. For $\bar{Y} := \min\{1, Y\}$

we have $E_P[\bar{Y}] > 0$. Then with $\theta := E_P[\bar{Y}] > 0$, setting $\varepsilon := \frac{\theta}{3}$ the choice $\delta := 1$ yields

$$|E_Q[\bar{Y}] - E_P[\bar{Y}]| \leq E_P|\frac{dQ}{dP}\bar{Y} - \bar{Y}| \leq E_P|\frac{dQ}{dP} - 1| < \frac{\theta}{3},$$

hence a lower bound is constructed for $E_Q[\bar{Y}]$, that is

$$E_Q[\bar{Y}] \geq E_P[\bar{Y}] - \frac{\theta}{3}.$$

Using this bound and the superhedging theorem we have the inequality for all $Q \in \mathcal{P}$ and Q-martingales $(Z_t)_{t\in[0,T]}$

$$0 \geq E_Q(\bar{Y}) - E_Q\int_0^T |Z_t - S_t|^2 dt > E_Q[\bar{Y}] - \frac{\theta}{3} \geq \frac{\theta}{3},$$

a contradiction which completes the proof.

□

In M. Rasonyi and P. Guasoni [2] it has been shown that a broad class of modells are enjoying the (NA2) property. To identify this class we shall introduce the notion of conditional full support. For this purpose we follow M. Rasonyi and P. Guasoni [2, Definition 4.3.]. Let μ denote the law of a stochastic process $S : \Omega \mapsto C_x[0,T]$, where $C_x[0,T]$ denotes the set of continuous functions on the domain $[0,T]$ with an initial value x. So the set function μ is a probability measure on $C_x[0,t]$. The support of the measure μ is defined and denoted by $supp(\mu) := \bigcap\{F : F \textit{ is closed}, F \subset C_x[0,T], \mu(F) = 1\}$.

Definition 2.14 (Conditional full support)**.** *We say that the process S_t has conditional full support in $\mathbb{R}$ if for all $t \in [0,T]$*

$$supp(P(S|_{[t,T]} \in \cdot|\mathcal{F}_t)) = C_{S_t}[t,T]$$

is satisfied.

In M. Rasonyi and P. Guasoni [2, Theorem 4.4] it is stated that in a market model with friction satisfying the superlinearity conditions in (2.1) if a process S_t has conditional full support then (NA2) holds.

Theorem 2.15. *If the process S_t has conditional full support then for all $\varepsilon > 0$ there exist a probability measure Q equivalent to P and a Q-martingale $(Z_t)_{t\geq 0}$ such that for all $t \geq 0$*

$$|S_t - Z_t| < \varepsilon. \tag{10}$$

The next theorem introduces the sufficient condition that identifies the class enjoying the (NA2) property. For comparison, note that this theorem requires a much stronger assumption then Theorem 2.12, since, loosely speaking, the theorem below states that the process S_t must be approximated by a martingale uniformly, and this - showing the strength of Theorem 2.12 - implies (8).

Theorem 2.16. *If for all $\varepsilon > 0$ there exists a probability measure $Q \in \mathcal{P}$ and a Q-martingale $(Z_t)_{t\geq 0}$ such that $E_Q|Z_T|^{\frac{\beta}{\beta-1}} < \infty$, where β is as in (5), such that*

$$\sup_{t>0} |S_t - Z_t| < \varepsilon, \tag{11}$$

almost surely, then the condition (NA2) holds.

Proof. Fixing $\varepsilon > 0$ and $(Z_t)_{t\in[0,T]}$ such that $\sup_{t>0} |S_t - Z_t| < \sqrt{\varepsilon/T}$, we have

$$E_Q \int_0^T |Z_t - S_t|^2 dt \leq T E_Q \sup_{t>0} |S_t - Z_t|^2 = T E_Q \big(\sup_{t>0} |S_t - Z_t|\big)^2 < \varepsilon,$$

and since Theorem 2.12 is applicable the proof is complete.

□

Thus, the class of processes with conditional full support possesses the (NA2) property by Theorems 2.15 and 2.16.

As an example, $W_t + g_t$ where $(W_t)_{t\in[0,1]}$ is a Brownian motion on $[0,1]$ and g_t is a continuous and deterministic function on $[0,1]$ with $g(0) = 0$ has conditional full support. Setting $g_t = 2\sqrt{t}$ by the Girsanov theorem one can show that in a frictionless market there is no equivalent martingale measure, so an arbitrage opportunity is generated with the price process set to $S_t := W_t + 2\sqrt{t}$. In contrast, in a model equipped with friction the same price process yields the condition (NA2), by the conditional full support property.

3. A process with property (NA2)

Now we will construct a discrete time process that satisfies the condition (NA2) in the superlinear friction case, but at the same time an arbitrage strategy can be constracted in the frictionless and transaction cost models.

For simplicity, the time horizon will be set to $T = 1$. We denote by $\mathbb{N}$ the positive integers, i.e. $\mathbb{N} := \{1, 2, ...\}$. Let $(\xi_i)_{i\in\mathbb{N}}$ be an independent sequence of discrete random variables, with distribution $P(\xi_i = 2) = 1 - P(\xi_i = \frac{1}{2}) = \frac{1}{2^i}$ for $i \in \mathbb{N}$. The price process is constructed through the following cumulative product generated by the sequence $(\xi_i)_{i\in\mathbb{N}}$. Let $X_0 = 1$ and for $k \in \mathbb{N}$

$$X_k := \prod_{i=1}^{k} \xi_i.$$

Now let $t_0 = 0$ and $(t_i)_{i\in\mathbb{N}}$ be a partition of the unit interval, i.e. $0 < t_1 < ... < t_i < ... < 1$. The partition will be explicitly given later, so that it can be chosen appropriately. We define the stock price process as

$$S_t := \sum_{k=0}^{\infty} X_k \mathbb{1}_{[t_k,t_{k+1})}(t).$$

For the process S_t we construct a right continuous filtration $(\mathcal{G}_t)_{t\geq 0}$ as follows. Set $\mathcal{G}_t$ trivial for $t \in [0, t_1)$ and for $0 < t_1 \leq t < 1$ let $\mathcal{G}_t := \sigma(\xi_i, 1 \leq i \leq k)$ if $t \in [t_k, t_{k+1})$, further let $\mathcal{G}_1 = \sigma(\xi_i, i \geq 1)$.

The process S_t starts from $S_0 = 1$ and, although it possibly could happen, S_t rarely jumps upwards and rapidly goes to zero after the first few time segments. The process X_n – and simultaneously the price driven by it – disappears almost surely as $n \to \infty$, a fact that is formulated by the next lemma.

Lemma 3.1. *The process X_k goes to zero almost surely as $k \to \infty$.*

Proof. Since $\sum_{i=0}^{\infty} \frac{1}{2^i} = 2 < \infty$ by the Borell-Cantelli Lemma we have that $P(\{\xi_i = 2\} \text{ infinitely often}) = 0$ that is there exists an almost surely finite random threshold $i_0 = i_0(\omega) < \infty$ such that if $i \geq i_0$ then $\xi_i = \frac{1}{2}$, so the equality

$$\lim_{k\to\infty} X_k(\omega) = \lim_{k\to\infty} \prod_{i=0}^{k} \xi_i(\omega) = \prod_{i=0}^{i_0(\omega)} \xi_i(\omega) \cdot \lim_{i\to\infty} \frac{1}{2^i} = 0$$

holds, by which the proof is complete. □

Remark 3.2. *Note that while convergence to zero is almost sure the process can reach arbitrary high values, i.e. for all fixed $N \in \mathbb{N}$, $P(\sup_{k\in\mathbb{N}} X_k > N) > 0$ holds.*

Now we prove that with an appropriately chosen partition, the sufficient condition in Theorem 2.12 is satisfied by S_t, so with superlinear friction the condition (NA2) holds.

Theorem 3.3. *With appropriately chosen sequence $(t_i)_{i\in\mathbb{N}}$, for all $\varepsilon > 0$ a measure $Q = Q^{(\varepsilon)}$ equivalent to P and a Q-martingale $Z_t = Z_t^{(\varepsilon)}$ can be constructed so that*

$$E_Q \int_0^1 |Z_t - S_t|^2 dt < \varepsilon. \tag{12}$$

Proof. We start with fixing an index $n \in \mathbb{N}$. Then Fubini's theorem and splitting up the Lebesgue integral in (12) yields

$$E_Q \int_0^1 |Z_t - S_t|^2 dt = \int_0^{t_n} E_Q |Z_t - S_t|^2 dt + \int_{t_n}^1 E_Q |Z_t - S_t|^2 dt. \tag{13}$$

Now we construct a measure Q_n equivalent to P such that the process is a Q_n-martingale on $[0, t_n]$. Equation (13) gives us the indication that Z_t is beneficial to be defined for $t \in [0, 1]$ as

$$Z_t = Z_t^{(n)} := S_t \mathbb{1}_{t\in[0,t_n)} + S_{t_n} \mathbb{1}_{t\in[t_n,1]}.$$

This way the integral (13) becomes

$$E_{Q_n} \int_0^1 |Z_t - S_t|^2 dt = \int_{t_n}^1 E_{Q_n} |S_{t_n} - S_t|^2 dt = \int_{t_n}^1 E \frac{dQ_n}{dP} |S_{t_n} - S_t|^2 dt, \tag{14}$$

where $\frac{dQ_n}{dP}$ denotes the Radon-Nikodym derivative of the measure Q_n with respect to P, for which we will construct an upper bound first. We know that the measure Q_n has to possess the property that S_t is a martingale with respect to it on the interval $[0, t_n]$. For the index $k \leq n$ let us assume that $0 < s < t$ is such that $0 < t_{k-1} \leq s < t_k \leq t < t_{k+1} < t_n$. Then using the definition of S_t and the martingale property we have that

$$\begin{aligned} &E_{Q_n}(S_t|\mathcal{G}_s) = S_s \Leftrightarrow E_{Q_n}(X_k|\sigma(\xi_1, ..., \xi_{k-1})) = X_{k-1} \\ &\quad \Leftrightarrow X_{k-1}E_{Q_n}(\xi_k) = X_{k-1}. \end{aligned}$$

This means that the measure Q_n has to satisfy $E_{Q_n}(\xi_k) = 1$ which yields the equation

$$2Q_n(\xi_k = 2) + \frac{1}{2}Q_n(\xi_k = \frac{1}{2}) = 1.$$

That is satisfied if and only if $Q_n(\xi_k = 2) = \frac{1}{3}$ and $Q_n(\xi_k = 2) = \frac{2}{3}$ for all $k \leq n$. Now by the nature of the model the measures Q_n and P are atomic, and as the Radon-Nikodym derivative is $\mathcal{G}_{t_n}$-measurable if $\omega \in \{\xi_1 = a_1, ..., \xi_n = a_n\}$ we have

$$\frac{dQ_n}{dP}(\omega) = \frac{Q_n(\{\xi_1 = a_1, ..., \xi_n = a_n\})}{P(\{\xi_1 = a_1, ..., \xi_n = a_n\})}.$$

Now the numerator $Q_n\{\xi_1 = a_1, , ..., \xi_n = a_n\}$ is bounded by 1, and the denomirator can be bounded from below as

$$\begin{aligned} &P(\{\xi_1 = a_1, ..., \xi_n = a_n\}) = \\ &\prod_{i=1}^{n} \Big(\frac{1}{2^i}\Big)^{\mathbb{1}_{\{\xi_i=2\}}} \Big(1 - \frac{1}{2^i}\Big)^{1-\mathbb{1}_{\{\xi_i=2\}}} > \Big(\frac{1}{2^{n+1}}\Big)^n. \end{aligned}$$

Resulting in an upper bound for the Randon-Nikodym derivative, that is

$$\frac{dQ_n}{dP} < \Big(2^{n+1}\Big)^n.$$

Direct calculations show that if $k \geq 3$ then $ES^2_{t_k} \geq ES^2_{t_{k+1}}$ and consequently for $n \geq 3$ and $t \geq t_n$ we have $ES^2_t \leq ES^2_{t_n}$. So for large enough n we have that $E|S_{t_n} - S_t|^2 \leq ES^2_{t_n} + ES^2_t \leq 2ES^2_{t_n} = 2EX^2_n \leq 2^{2n+1}$. Using this the integral in (14) can be bounded as

$$\int_{t_n}^{1} E\frac{dQ_n}{dP}|S_{t_n} - S_t|^2 dt \leq \big(2^{n+1}\big)^n 2^{2n+1}(1 - t_n) = 2^{n^2+3n+1}(1 - t_n).$$

Setting $t_i := 1 - i^{-1}2^{-(i^2+3i+1)}$, yields the bound $\frac{1}{n}$ for the integral in (14). So for any $\varepsilon > 0$ we can choose n large enough such that Q_n and $Z_t = Z^{(n)}_t$ is such that $E_Q \int_0^1 |Z_t - S_t|^2 dt < \varepsilon$, and the proof is complete. □

Now, on one hand, note that using the bound for the Radon-Nikodym derivative

$$E_{Q_n}[Z^3_T] = E_{Q_n}[S^3_{t_n}] = E_{Q_n}[X^3_n] = E[\frac{dQ_n}{dP}X^3_n] \leq (2^{n+1})^n 2^{3n} < \infty. \tag{15}$$

On the other hand, the measure Q_n is in class $\mathcal{P}$, see (5), by the following reason. Using Fubini's theorem and the bound for the Radon-Nikodym derivative

(16)

$$E_{Q_n}\int_0^1(1+|S_t|)^6dt=\int_0^1 E_{Q_n}(1+|S_t|)^6dt\leq(2^{n+1})^n\int_0^1(1+E|S_t|)^6dt,$$

and independence of the ξ_i yields

$$\sup_t E[S_t^6]\leq\sup_k E[X_k^6]=\sup_k\prod_{i=1}^k E[\xi_i^6]$$
$$\leq\sup_k\prod_{i=1}^k(\frac{1}{2^i}2^6+\frac{1}{2})\leq\sup_{k\leq 7}\prod_{i=1}^k(\frac{1}{2^i}2^6+\frac{1}{2})<\infty.$$

This means that the expression $E_{Q_n}\int_0^1(1+|S_t|)^6dt$ is finite for all $n\in\mathbb{N}$, hence $Q_n\in\mathcal{P}$ holds.

Using Theorem 3.3, (15) and (16) the hypothesis of Theorem 2.12 are satisfied and it is proved that the price process S_t generates no arbitrage opportunity of the second kind.

In summary, we constructed a process S_t that yields no opportunities for arbitrage of the second kind with superlinear friction, but an arbitrage strategy can be created in a model where one can perform trading with infinite speed.

Our price process initiates at value one and we manage to go short with one piece of share somewhere at the beggining. The process has the particular property that cutting it somewhere near the time horizon with a measure of change it behaves as a martingale, so no matter what the model is, we have no chance of beating the system. But infinitesimally close to time $T=1$ the price can be exploited to make riskless profit.

When we arrive at our time horizon the share worths nothing so we pay back nothing. That being said if we manage to get rid of our one share during the infinitesimally small time period at the end, we make money without risk, and an opportunity of arbitrage has been constructed. Further note, that transaction costs do not change this fact, the arbitrage opportunity is still present.

It will take time to buy this one piece of share, and also getting rid of it. That is of no worry in a frictionless market (or in a model with transaction costs) but with superlinear friction such a fast paced activity is punished with unbounded losses. This observation spotlights an important difference between the models in terms of arbitrage of the second kind.

4. A class with property (NA)

In this chapter we wish to find a wider class of processes that can be employed as example for a model with the (NA) property. Note that as (NA) is a stronger assumption compared to (NA2), in the sense that if (NA) holds so does (NA2), in this model not only arbitrage of the first kind is absent, but the same can be stated for arbitrage of the second kind. Still, the same price process generates arbitrage opportunity both with transaction costs and in the frictionless case.

We will construct a price process that can not be approximated with a martingale uniformly. This can be achieved by creating a process that acts as a martingale under an appropriate equivalent measure up until a certain time close to a finite time horizon, and after that it rapidly goes to zero. We will utilize this idea here again.

The well-known identity

$$E_Q[\xi|\mathcal{F}] = \frac{E[\frac{dQ}{dP}\xi|\mathcal{F}]}{E[\frac{dQ}{dP}|\mathcal{F}]} \tag{17}$$

will be used for changing measure under conditional expectation, where ξ is a random variable on some probability space $(\Omega, \mathcal{A}, P)$ equipped with another measure Q equivalent to P and a sub-σ-algebra $\mathcal{F}$, such that these expectations exist.

Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed discrete random variables, such that ξ_i is concentrated on integer values and $P(\xi_i = k) > 0$ for all $k \in \mathbb{Z}$ and further assume that the sixth moment of the ξ_i exists, i.e. $E[\xi_1^6] < \infty$ and we also assume $E[\xi_1] = 0$.

Remark 4.1. *We note here that, with little modification and more complex notation, one can initiate the construction given here without assuming zero mean and identical distribution for the ξ_i. Further the finite sixth moment assumption can also be weakened, i.e. for $\delta > 0$ it can be achieved that $E[\xi_i^{2+\delta}] < \infty$ is sufficient. Even though we are not pursuing such generality here, with the tools given below, the results can be put in a more general framework with minor effort.*

Let $X_0 := 1$ and let $(X_i)_{i\in\mathbb{N}}$ denote the sequence of the partial sums scaled down by a factor that ensures that the process goes to zero almost surely. We explicitly define

$$X_k := \frac{1 + \xi_1 + ... + \xi_k}{k}. \tag{18}$$

Almost sure convergence to zero can be seen easily, due to finite second moment, by using the law of lare numbers, that states the almost sure convergence of the mean $k^{-1}(\xi_1 + ... + \xi_k)$ to the expected value $E\xi_1 = 0$.

Also equip the space with the filtration $(\mathcal{F}_i)_{i\in\mathbb{N}}$ defined by $\mathcal{F}_k = \sigma(\xi_1, ..., \xi_k)$ for $k \geq 1$, and also set $\mathcal{F}_0$ trivial.

We will construct an equivalent measure $\bar{Q} = \bar{Q}_n$ such that the stopped process $(X_i)_{i=0}^n$ is a martingale with respect to it. For this to happen one would require that

$$E_{\bar{Q}}[X_k|\mathcal{F}_{k-1}] = X_{k-1}, \tag{19}$$

for all $k \leq n$. It turns out that this problem can be reduced to constructing an equivalent measure Q_k for fixed $k \leq n$ that has the property $E_{Q_k}[X_k|\mathcal{F}_{k-1}] = X_{k-1}$, see the details below.

By the identity (17) our requirement becomes

$$\frac{E[\frac{dQ_k}{dP}X_k|\mathcal{F}_{k-1}]}{E[\frac{dQ_k}{dP}|\mathcal{F}_{k-1}]} = X_{k-1}.$$

Hence it is sufficient to have

$$E[\frac{dQ_k}{dP}X_k|\mathcal{F}_{k-1}] = X_{k-1} \quad \text{and} \quad E[\frac{dQ_k}{dP}|\mathcal{F}_{k-1}] = 1. \tag{20}$$

The Radon-Nikodym derivative $\frac{dQ_k}{dP}$ will be a function of the random variables $\xi_1, ..., \xi_k$ so we will look for it in the form $f(\xi_1, ..., \xi_k)$ for some measurable function $f_k : \mathbb{R}^k \mapsto \mathbb{R}$. So the existence of the measure Q_k boils down to the search for the function f_k. This way the requirements can be rewritten as

$$E[f_k(\xi_1, ..., \xi_k)X_k|\mathcal{F}_{k-1}] = X_{k-1} \quad \text{and} \quad E[f_k(\xi_1, ..., \xi_k)|\mathcal{F}_{k-1}] = 1, \tag{21}$$

which are - after some algebraic manipulations - equivalent to

$$E[f_k(\xi_1, ..., \xi_k)\xi_k|\mathcal{F}_{k-1}] = g(\xi_1, ...\xi_{k-1}) \quad \text{and} \quad E[f_k(\xi_1, ..., \xi_k)|\mathcal{F}_{k-1}] = 1, \tag{22}$$

for some function $g = g_{k-1} : \mathbb{R}^{k-1} \mapsto \mathbb{R}$. In our example, the function $g_{k-1}(\xi_1, ..., \xi_{k-1})$ will, in fact, be a linear combination of the random variables $\xi_1, ..., \xi_{k-1}$ with some additive constant. So finally, using measurability properties, we will require

$$E[f_k(x_1, ..., x_{k-1}, \xi_k)[\xi_k - g_{k-1}(x_1, ...x_{k-1})]] = 0, \tag{23}$$

and

$$E[f_k(x_1, ..., x_{k-1}, \xi_k)] = 1 \tag{24}$$

to hold for all possible values $x_1, ..., x_{k-1}$.

It is also possible to construct this new equivalent measure Q_k in such a way that the total variation norm $||P - Q_k||_{tv}$ is small. For the process X and for fixed $k \leq n$ the next lemma shows the existence of such measure.

Lemma 4.2. *Let ϑ be a discrete random variable concentrated on integer values such that $P(\vartheta = l) > 0$ for all $l \in \mathbb{Z}$. Under these assumptions, for all fixed $\varepsilon > 0$ and $k \geq 1$ there exists a function $f = f_k = f(k, \varepsilon) : \mathbb{R}^k \mapsto \mathbb{R}$ such that*

$$E[f_k(x_1, ..., x_{k-1}, \vartheta)[\vartheta - g_{k-1}(x_1, ...x_{k-1})]] = 0,$$

and

$$E|f_k(x_1, ..., x_{k-1}, \vartheta) - 1| < \varepsilon$$

holds for all possible values $x_1, ..., x_{k-1}$, where $g = g_{k-1}$ is as in (23).

Proof. Fix $\varepsilon > 0$ and $x_1, ..., x_{k-1} \in \mathbb{Z}$. Let $l \in \mathbb{Z}$ and denote $p_l := P(\vartheta = l)$ and $M := E\vartheta$. Define $b = b_{k-1}(l_0, x_1, ..., x_{k-1}) := \frac{g_{k-1}(x_1,...,x_{k-1})-M}{l_0-g_{k-1}(x_1,...,x_{k-1})}$ for some $l_0 = l_0^{(k-1)} \in \mathbb{Z}$, depending on $(x_1, ..., x_{k-1})$, so that $0 < b < \varepsilon/2$. Note that b, in fact, can be chosen to satisfy the previous requirement, since the variable l_0 depending on $(x_1, ..., x_{k-1})$ can be chosen arbitrary large in absolute value. Set $f_k(x_1, ..., x_{k-1}, l) := \frac{1}{1+b}$ if $l \neq l_0$ and $f_k(x_1, ..., x_{k-1}, l_0) := \frac{p_{l_0}+b}{p_{l_0}(1+b)}$. Then

$$
\begin{aligned}
&E[f(x_1, ..., x_{k-1}, \vartheta)[\vartheta - g(x_1, ..., x_{k-1})]] \\
&\quad = \sum_{l \neq l_0} p_l f(x_1, ..., x_{k-1}, l)[l - g(x_1, ..., x_{k-1})] \\
&\quad + p_{l_0} f(x_1, ..., x_{k-1}, l_0)[l_0 - g(x_1, ..., x_{k-1})] \\
&\quad = \sum_{l \neq l_0} \frac{1}{1+b} p_l[l - g(x_1, ..., x_{k-1})] + \frac{p_{l_0}+b}{(1+b)}[l_0 - g(x_1, ..., x_{k-1})] \\
&\quad = \frac{1}{1+b}\Big[\sum_{l \neq l_0} p_l[l - g(x_1, ..., x_{k-1})] + (p_{l_0}+b)[l_0 - g(x_1, ..., x_{k-1})]\Big] \\
&\quad = \frac{1}{1+b}\Big[\sum_{l \in \mathcal{R}} p_l[l - g(x_1, ..., x_{k-1})] + b[l_0 - g(x_1, ..., x_{k-1})]\Big] \\
&\quad = \frac{1}{1+b}\Big[M - g(x_1, ..., x_{k-1}) + g(x_1, ..., x_{k-1}) - M\Big] = 0
\end{aligned}
$$

and also

$$
\begin{aligned}
&E|f(x_1, ..., x_{k-1}, \vartheta) - 1| \\
&\quad = \sum_{l \neq l_0} p_l |f(x_1, ..., x_{k-1}, l) - 1| + p_{l_0}|f(x_1, ..., x_{k-1}, l) - 1| \\
&\quad = \sum_{l \neq l_0} p_l \Big|\frac{1}{1+b} - 1\Big| + \Big|\frac{b(1-p_{l_0})}{1+b}\Big| \\
&\quad \leq \Big|\frac{1}{1+b} - 1\Big| + \Big|\frac{b}{1+b}\Big| = 2\Big|\frac{b}{1+b}\Big| < \varepsilon,
\end{aligned}
$$

and the proof is complete. □

Remark 4.3. *Later, a bound will be needed for $|l_0|$ which we will construct now. First note that we can define the value l_0 explicitly for $\vartheta = \xi_i$. As $M = E[\xi_i]$ is zero in our case, we have for the variable b*

$$b = \frac{g_{k-1}}{l_0 - g_{k-1}} < \varepsilon/2.$$

Here we have to consider the cases $g_{k-1} > 0$ and $g_{k-1} < 0$ separately. Note that in the case of positive g_{k-1}, $l_0 - g_{k-1}$ is positive also and in the case of negative g_{k-1}, $l_0 - g_{k-1}$ is negative likewise. Using this, algebraic *manipulation yields the inequalities*

$$(1+\frac{2}{\varepsilon})g_{k-1} < l_0, \quad and \quad (1+\frac{2}{\varepsilon})g_{k-1} > l_0$$

respectively. So in the case $g_{k-1} > 0$*, we choose* $l_0 := \lfloor (1+\frac{2}{\varepsilon})g_{k-1} - 1 \rfloor$ *and in the case* $g_{k-1} > 0$*, we choose* $l_0 := \lceil (1+\frac{2}{\varepsilon})g_{k-1} + 1 \rceil$*. Now note that* $\max\{|x-1|, |x+1|\} = |x| + 1$*, thus we can create the bound as*

$$|l_0| \leq (1+\frac{2}{\varepsilon})|g_{k-1}| + 2.$$

Now we present a theorem stating that, for arbitrary fixed $n \in \mathbb{N}$, the probability space can be altered by a new equivalent measure $\bar{Q}_n$ so that up until time n the process X behaves as a martingale, furthermore, the new measure is arbitrarily close to the original measure in the sense of total variation.

We will utilize Lemma 4.2. The proof of the theorem merely relies on the fact that the process X is built up by independent components so that one can treat the process with reweighting the probability space with the new measure step by step. It also employs the idea that the alteration of the new measure can be arbitrarily small due to the specific property of the independent building blocks, that is, they are unbounded from above and below.

Theorem 4.4. *Let the process X be defined as above and fix an index* $n \in \mathbb{N}$*. There exists a measure* $\bar{Q}_n$ *equivalent to* P *such that* $(X_i)_{i=0}^n$ *is a martingale with respect to this measure, and for any fixed* $\varepsilon > 0$ *the total variation norm* $||P - \bar{Q}_n||_{tv} < \varepsilon$*.*

Proof. Let us fix $n \in \mathbb{N}$ and define $\frac{d\bar{Q}_n}{dP} := \frac{dQ_1}{dP} \cdots \frac{dQ_n}{dP}$, where for $0 < k \leq n$ the existence of $\frac{dQ_k}{dP}$ is guaranteed by Lemma 4.2, i.e. $\frac{dQ_k}{dP} = f_k(\xi_1, ..., \xi_k)$, where the function f_k is as in (23) and (24). Throughout this proof we mostly operate with the well-known tower property of conditional expectation. First we show that $\frac{d\bar{Q}_n}{dP}$ integrates to 1. The first step - using that $E[f_n(\xi_1, ..., \xi_n)|\mathcal{F}_{n-1}] = 1$ by the construction of f_n - goes as

$$\begin{aligned} E[\frac{d\bar{Q}_n}{dP}] =& E\Big[E[\frac{d\bar{Q}_n}{dP}|\mathcal{F}_{n-1}]\Big] = E\Big[\frac{dQ_1}{dP} \cdots \frac{dQ_{n-1}}{dP} E[f_n(\xi_1, ..., \xi_n)|\mathcal{F}_{n-1}]\Big] \\ &= E\Big[\frac{dQ_1}{dP} \cdots \frac{dQ_{n-1}}{dP}\Big]. \end{aligned}$$

Iterating this yields the result, that is $E[\frac{d\bar{Q}_n}{dP}] = 1$. Then attempting to trace down the martingale property, for $k \leq n$ one first encounters the equalities

$$E_{\bar{Q}_n}[X_k|\mathcal{F}_{k-1}] = \frac{E[f_n \cdots f_1 X_k|\mathcal{F}_{k-1}]}{E[f_n \cdots f_1|\mathcal{F}_{k-1}]} = \frac{E[f_n \cdots f_k X_k|\mathcal{F}_{k-1}]}{E[f_n \cdots f_k|\mathcal{F}_{k-1}]}.$$

Again by the tower property and by the construction of the measures, one has for the denominator

$$E[f_n \cdots f_k|\mathcal{F}_{k-1}] = E[E[f_n \cdots f_k|\mathcal{F}_{n-1}]|\mathcal{F}_{k-1}]$$
$$= E[f_{n-1} \cdots f_k E[f_n|\mathcal{F}_{n-1}]|\mathcal{F}_{k-1}] = E[f_{n-1} \cdots f_k|\mathcal{F}_{k-1}].$$

By iterating this step we can see that the denominator is 1. Using this, we treat the numerator

$$E_{\bar{Q}_n}[X_k|\mathcal{F}_{k-1}] = E[f_n \cdots f_k X_k|\mathcal{F}_{k-1}]$$

as before. Equations (21) and iteration yields

$$E[f_n \cdots f_k X_k|\mathcal{F}_{k-1}] = E[E[f_n \cdots f_k X_k|\mathcal{F}_{n-1}]|\mathcal{F}_{k-1}]$$
$$= E[f_{n-1} \cdots f_k X_k E[f_n|\mathcal{F}_{n-1}]|\mathcal{F}_{k-1}]$$
$$= E[f_{n-1} \cdots f_k X_k|\mathcal{F}_{k-1}] = E[f_{n-1} \cdots f_k X_k|\mathcal{F}_{k-1}]$$
$$= E[f_k X_k|\mathcal{F}_{k-1}] = X_{k-1},$$

and this exactly means that the process X is a martingale until time n. Now we fix $\varepsilon > 0$ and consider the total variation norm $||P - \bar{Q}_n||_{tv} = E|\frac{d\bar{Q}_n}{dP} - 1|$. Some algebraic manipulations yield

$$E\Big|\frac{dQ_1}{dP} \cdots \frac{dQ_n}{dP} - 1\Big| = E\big|f_1 \cdots f_n - f_2 \cdots f_n + f_2 \cdots f_n - 1\big|$$
$$\leq E\Big[\big|f_1 - 1\big|f_2 \cdots f_n\Big] + E\big|f_2 \cdots f_n - 1\big|$$

Then using the tower property again

$$E\Big[\big|f_1 - 1\big|f_2 \cdots f_n\Big]$$
$$= E\Big[E\Big[\big|f_1 - 1\big|f_2 \cdots f_n\big|\mathcal{F}_{n-1}\Big]\Big] = E\Big[\big|f_1 - 1\big|f_2 \cdots f_{n-1}\Big].$$

Iterating this we get

$$E\Big|\frac{dQ_1}{dP} \cdots \frac{dQ_n}{dP} - 1\Big| \leq E\big|f_1 - 1\big| + E\big|f_2 \cdots f_n - 1\big|,$$

of which further iteration yields the upper bound

$$E\Big|\frac{dQ_1}{dP} \cdots \frac{dQ_n}{dP} - 1\Big| \leq \sum_{i=1}^{n} E\Big|\frac{dQ_i}{dP} - 1\Big|.$$

By Lemma 4.2 it is doable that we choose the Radon-Nikodym derivatives so that $E\big|\frac{dQ_i}{dP} - 1\big| < \frac{\varepsilon}{n}$ for $0 < i \leq n$, that completes the proof. □

Now we show that the sufficient condition for (NA) formulated in Theorem 2.16 is applicable. Just as in the previous example, let $t_0 = 0$ and $(t_i)_{i \in \mathbb{N}}$ be a partition of the unit interval, i.e. $0 = t_0 < t_1 < ... < t_n < ... < 1$. The partition will be explicitly given later, so that it can be chosen appropriately.

We define the stock price process as

$$S_t := \sum_{i=0}^{\infty} X_i \mathbb{1}_{[t_i, t_{i+1})}(t).$$

For the process S_t we construct a right continuous filtration $(\mathcal{G}_t)_{t\geq 0}$ as follows. Set $\mathcal{G}_t$ trivial for $t \in [0, t_1)$ and for $0 < t_1 \leq t < 1$ let $\mathcal{G}_t := \sigma(\xi_i, 1 \leq i \leq k)$ if $t \in [t_k, t_{k+1})$, further let $\mathcal{G}_1 = \sigma(\xi_i, i \geq 1)$. Note that the continuous filtration $\mathcal{G}_t$ at time $t \in [t_k, t_{k+1})$ coincides with $\mathcal{F}_k$, an element of the discrete filtration defined and used above.

Remark 4.5. *The price process S_t can take both negative and positive values. For various reasons it is not a problem. On one hand for a fixed distribution of the sequence $(\xi_i)_{i\in\mathbb{N}}$ with an appropriate linear transformation the probability that the price process takes negative values can be made arbitrarily small. On the other hand, certain financial products, for example future contracts, are modelled with real valued processes.*

Now we will shown that Theorem 2.13 is applicable. The process $(Z_t)_{t\in[0,1]}$ will be defined as

$$Z_t = Z_t^{(n)} := S_t \mathbb{1}_{[0,t_n)} + S_{t_n} \mathbb{1}_{[t_n,1]}, \tag{25}$$

and can be thought of as S_t frozen at time t_n. As previously shown in Theorem 4.4, there exists a measure $\bar{Q}_n$ such that up until time n the price process is itself a martingale with respect to the measure. On the interval $[0, t_n)$ we have $Z_t = S_t$ and thus the integrand disappears up until time $t = t_n$, so

$$E_{\bar{Q}_n} \int_0^1 |Z_t - S_t|^2 dt = 0 + \int_{t_n}^1 E_{\bar{Q}_n} |S_{t_n} - S_t|^2 dt. \tag{26}$$

On one hand, we want to bound the integrand $E_{\bar{Q}_n}|S_{t_n} - S_t|^2$. On the other hand, the expectation $E_{\bar{Q}_n}[|Z_T|^3] = E_{\bar{Q}_n}[|Z_1|^3] = E_{\bar{Q}_n}[|X_n|^3]$ has to be finite. Further, we also have to show that the measure $\bar{Q}_n$ is in class $\mathcal{P}$, for the definition of class $\mathcal{P}$ see (5). This last fact will be proven first, that is, we have to show that the integral in (5), $\int_0^1 E_{\bar{Q}_n}(1+|S_t|)^6 dt$ is finite, and for this it is sufficient that

$$E_{\bar{Q}_n}[(1 + |S_t|)^6] \leq 2^5 + 2^5 E_{\bar{Q}_n}[S_t^6]. \tag{27}$$

is bounded and integrable. Now, again, for some $m \geq 0$ it is true that $t \in [t_m, t_{m+1})$, and note that $X_0 = 1$. So it is enough to prove that $E_{\bar{Q}_n}[S_{t_m}^6] < \infty$ for all $m \geq 1$. We will show this for $m \geq n$, because the proof automatically generalizes to $m \geq 1$.

For a convex function $\varkappa : \mathbb{R} \to \mathbb{R}$, by the definition of convexity for fixed $l \in \mathbb{N}$ and $x_1, ..., x_l \in \mathbb{R}$ we have that

$$\varkappa(\frac{1}{l}\sum_{j=1}^{l} x_j) \leq \frac{1}{l}\sum_{j=1}^{l} \varkappa(x_j).$$

Using this for $\varkappa(x) = x^z$, where $z \in \mathbb{N}$ is even we have

$$(\sum_{j=1}^{l} x_j)^z \leq l^{z-1} \sum_{j=1}^{l} x_j^z. \tag{28}$$

So using (28), we have for $m \geq n$

$$\begin{aligned} E_{\bar{Q}_n}[S_{t_m}^6] &= E_{\bar{Q}_n}[X_m^6] \\ &= E_{\bar{Q}_n}[\big(\frac{1+\xi_1+...+\xi_m}{m}\big)^6] \\ &= E_{\bar{Q}_n}[\big(\frac{1+\xi_1+...+\xi_n}{m} + \frac{\xi_{n+1}+...+\xi_m}{m}\big)^6] \\ &\leq 2^5 E_{\bar{Q}_n}[\big(\frac{1+\xi_1+...+\xi_n}{m}\big)^6] + 2^5 E_{\bar{Q}_n}[\big(\frac{\xi_{n+1}+...+\xi_m}{m}\big)^6]. \end{aligned} \tag{29}$$

Using independence and using (28) again we treat the second expectation as

$$\begin{aligned} &E_{\bar{Q}_n}[\big(\frac{\xi_{n+1}+...+\xi_m}{m}\big)^6] = E[\frac{d\bar{Q}_n}{dP}\big(\frac{\xi_{n+1}+...+\xi_m}{m}\big)^6] \\ &= E[\frac{d\bar{Q}_n}{dP}]E[\big(\frac{\xi_{n+1}+...+\xi_m}{m}\big)^6] = E[\big(\frac{\xi_{n+1}+...+\xi_m}{m}\big)^6] \\ &\leq \frac{(m-n)^5 E[\xi_1^6]}{m^6} \leq E[\xi_1^6]. \end{aligned} \tag{30}$$

The expression $E_{\bar{Q}_n}[\big(\frac{1+\xi_1+...+\xi_n}{m}\big)^6]$, using $m \geq n \geq 1$ and identity (28), can be bounded as

$$E_{\bar{Q}_n}[\big(\frac{1+\xi_1+...+\xi_n}{m}\big)^6] \leq (n+1)^5(1 + E_{\bar{Q}_n}[\xi_1^6] + ... + E_{\bar{Q}_n}[\xi_n^6]). \tag{31}$$

Hence it is sufficient to show that for $1 \leq i \leq n$ the expectation $E_{\bar{Q}_n}[\xi_i^6]$ is finite. We do this by induction, using the construction of the Radon-Nikodym derivative $\frac{d\bar{Q}_n}{dP} = \frac{dQ_1}{dP} \cdots \frac{dQ_n}{dP}$ and the definition of the Radon-Nikodym derivatives $\frac{dQ_k}{dP} = f_k$, for $k \leq n$, where f_k is as in (23) and (24) and in Lemma 4.2. We will also use the notation $p_l := P(\xi_1 = l)$.

For i fixed we have

$$E_{\bar{Q}_n}[\xi_i^6] = E[\frac{d\bar{Q}_n}{dP}\xi_i^6] = E[f_1 \cdots f_n \xi_i^6],$$

and iterating the tower property and using (20) we have

$$E_{\bar{Q}_n}[\xi_i^6] = E[f_1 \cdots f_i \xi_i^6]. \tag{32}$$

Note also, that by Remark 4.3 for all $1 \leq k$ we have for $l_0 = l_0^{(k-1)}$ depending on $(x_1, ..., x_{k-1})$, that

$$l_0^6 \leq ((1 + \frac{2}{\varepsilon})|g_{k-1}| + 2)^6 \leq 2^5(1 + \frac{2}{\varepsilon})^6 g_{k-1}^6 + 2^{11}. \tag{33}$$

With the notation $c_1 := 2^5(1+\frac{2}{\varepsilon})^6$ and $c_2 := 2^{11}$ the above inequality becomes

$$l_0^6 \le c_1 g_{k-1}^6 + c_2. \tag{34}$$

As the first step of the inductive proof, for $i=1$, we use the construction of $f_1 = f_1(x_1)$ in the proof on Lemma 4.2. Observe, that for $i=1$ the variable $b=b_0$ is a nonrandom real number and $g=g_0=1$. Employing (32) and (34), we have

$$\begin{aligned}
&E_{\bar{Q}_n}[\xi_1^6] = E[f_1\xi_1^6] = E[f_1(\xi_1)\xi_1^6] \\
&= \sum_{l\in\mathcal{R}(\xi_1)} f_1(l)l^6 p_l = f_1(l_0)l_0^6 p_{l_0} + \sum_{l\neq l_0} f_1(l)l^6 p_l \\
&= \frac{p_{l_0}+b}{(1+b)} l_0^6 + \frac{1}{1+b}\sum_{l\neq l_0} l^6 p_l \le l_0^6 + E[\xi_1^6] \\
&\le c_1 + c_2 + E[\xi_1^6] < \infty.
\end{aligned}$$

Now, we establish the inductive hypothesis, that is, for the fixed index $1 \le i < n$ it is true that

$$E_{\bar{Q}_n}[\xi_i^6] < \infty.$$

It needs to be shown that $E_{\bar{Q}_n}[\xi_{i+1}^6] < \infty$ holds. Using (32), and measurability properties

$$E_{\bar{Q}_n}[\xi_{i+1}^6] = E[f_1\cdots f_{i+1}\xi_{i+1}^6] = E[f_1\cdots f_i E[f_{i+1}\xi_{i+1}^6|\mathcal{F}_i]]. \tag{35}$$

We employ the notation of Lemma 4.2 again, i.e. $f_{i+1} = f_{i+1}(x_1,...,x_{i+1})$, $g_i = g_i(x_1,...,x_i)$, $b=b_i=b_i(x_1,...,x_i)$, and the variable $l_0 = l_0^{(i)}$ depending on $(x_1,...,x_i)$ all are as in the proof of Lemma 4.2. Using the construction of $f_{i+1} = f_{i+1}(x_1,...,x_{i+1})$ and also employing (34) similarly as before, we give a non-deterministic upper bound for $E[f_{i+1}\xi_{i+1}^2|\mathcal{F}_i]$ as

$$\begin{aligned}
&E[f_{i+1}\xi_{i+1}^6|\mathcal{F}_i] = E[f_{i+1}(\xi_1,...,\xi_i,\xi_{i+1})\xi_{i+1}^6|\mathcal{F}_i] \\
&\quad = \sum_{l\in\mathcal{R}_1} f_{i+1}(\xi_1,...,\xi_i,l)l^6 p_l \\
&\quad = f_{i+1}(\xi_1,...,\xi_i,l_0)l_0^6 p_{l_0} + \sum_{l\neq l_0} f_{i+1}(\xi_1,...,\xi_i,l)l^6 p_l \\
&\quad = \frac{p_{l_0}+b}{(1+b)} l_0^6 + \sum_{l\neq l_0}\frac{1}{1+b} l^6 p_l \le \frac{p_{l_0}+b}{(1+b)} l_0^6 + \frac{1}{1+b}\sum_{l\in\mathcal{R}_1} l^6 p_l \\
&\quad \le l_0^6 + E[\xi_1^6] \le c_1 g_i^6 + c_2 + E[\xi_1^6]
\end{aligned}$$

Using this and (35) the expression $E_{\bar{Q}_n}[\xi_{i+1}^6]$ can be bounded as

$$E_{\bar{Q}_n}[\xi_{i+1}^6] \le c_1 E[f_1\cdots f_i g_i(\xi_1,...,\xi_i)^6] + c_2 + E[\xi_1^6].$$

Since we know that the function g is just a linear combination of its variables plus a constant, using (28) again, with appropriate constants $a_0, a_1, ..., a_i \in$

$\mathbb{R}$ we have $g_i(\xi_1,...,\xi_i)^6 \leq (i+1)^5(a_0^6 + a_1^6\xi_1^6 + ... + a_i^6\xi_i^6)$. Hence the sixth moment of ξ_{i+1} under the measure $\bar{Q}_n$ can be bounded by

$$E_{\bar{Q}_n}[\xi_{i+1}^6] \leq E[\xi_1^6] + c_2 + c_1(i+1)^5\big(a_0^6 + \sum_{j=1}^{i} a_j^6 E[f_1 \cdots f_i \xi_j^6]\big).$$

This last inequality, formula (32) and the inductive hypothesis completes the proof, that is, we have for all $1 \leq i \leq n$ that

$$E_{\bar{Q}_n}[\xi_i^6] < \infty.$$

By this last result there exists a constant $K_n < \infty$, depending on n, so that

$$1 + E_{\bar{Q}_n}[\xi_1^6] + ... + E_{\bar{Q}_n}[\xi_n^6] < K_n. \tag{36}$$

By (29), (30), (31) and (36) above $E_{\bar{Q}_n}[S_t^6]$ can be bounded by

$$E_{\bar{Q}_n}[S_t^6] < 2^5(E[\xi_1^6] + (n+1)^5 K_n) =: \bar{K}_n. \tag{37}$$

Using this the inequality (27) becomes

$$E_{\bar{Q}_n}[(1+|S_t|)^6] < 2^5 + 2^5\bar{K}_n.$$

This implies that

$$E_Q \int_0^1 (1+|S_t|)^6 dt < \infty, \tag{38}$$

and this precisely means that the measure $\bar{Q}_n$ is in calss $\mathcal{P}$.

Note that for $1 \leq q < 6$ and $m > 1$ it is true that

$$E_{\bar{Q}_n}[|X_m|^q] \leq 1 + E_{\bar{Q}_n}[X_m^6] < 1 + \bar{K}_n < \infty. \tag{39}$$

This way the finiteness of $E_{\bar{Q}_n}[|Z_T|^3]$ is trivial,i.e

$$E_{\bar{Q}_n}[|Z_T|^3] = E_{\bar{Q}_n}[|Z_1|^3] = E_{\bar{Q}_n}[|X_n|^3] < \infty. \tag{40}$$

The expectation $E_{\bar{Q}_n}|S_{t_n} - S_t|^2$ can be bounded as follows. For all $t \in [t_n, 1]$ there exists some $m \geq n$ so that $t \in [t_m, t_{m+1})$, hence we have $S_t = X_m$, so for this m

$$E_{\bar{Q}_n}|S_{t_n} - S_t|^2 \leq 2E_{\bar{Q}_n}[X_n^2] + 2E_{\bar{Q}_n}[X_m^2].$$

Using (39) the expectation $E_{\bar{Q}_n}|S_{t_n} - S_t|^2$ is finite and the integral in (26) can be bounded by

$$\int_{t_n}^1 E_{\bar{Q}_n}|S_{t_n} - S_t|^2 dt \leq (1-t_n)4(1+\bar{K}_n).$$

Now we require $\int_{t_n}^1 E_{\bar{Q}_n}|S_{t_n} - S_t|^2 dt \leq \frac{1}{n}$ to hold, then the partition $(t_i)_{i\in\mathbb{N}}$ can be defined for $i \geq 1$ as

$$t_i := 1 - \frac{1}{i4(1+\bar{K}_i)}.$$

So with this partition for fixed $\varepsilon > 0$, n can be chosen so large that with $\bar{Q}_n$ and $Z_t = Z_t^{(n)}$ constructed above $E_{\bar{Q}_n} \int_0^1 |Z_t - S_t|^2 dt < \varepsilon$ holds and $E|\frac{d\bar{Q}_n}{dP} - 1| < \varepsilon$ is satisfied also. By this the following theorem is proved.

Theorem 4.6. *Choose the sequence $(t_i)_{i\in\mathbb{N}}$ as above, then for all $\varepsilon > 0$ a measure $Q = Q(\varepsilon)$ equivalent to P and a Q-martingale $(Z_t = Z_t(\varepsilon))_{t\geq 0}$ can be constructed so that*

$$E_Q \int_0^1 |Z_t - S_t|^2 dt < \varepsilon, \quad \textit{and} \quad ||P - Q||_{tv} < \varepsilon.$$

With (38), (40) and Theorem 4.6 it is clear that Theorem 2.13 is applicable, hence it can be stated that in a model with superlinear friction the price S_t generates no arbitrage opportunity of the first kind. In contrast, without superlinear friction, a simple strategy can be created, see for example (4), that exploits the price without risk, that is to say arbitrage is present in the two other models, i.e. in the transaction cost model and in the frictionless case. This shows that in illiquid markets, i.e. where superlinear friction is present, it is indeed more difficult to create arbitrage as opposed to markets where only transaction costs are present.

5. Conclusions

In this study we have constructed examples which show that in illiquid markets (where trading costs are a superlinear function of trading speed) it is more difficult to create arbitrage then in models where transaction costs are linear (or there are no such costs at all). We have achived this by constructing price processes which allow riskless profit if trades can be executed infinitely fast but fininte speed misses these opportunities.

Our results help to clarify differences between various trading mechanisms in terms of the theory of arbitrage.

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